NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 22ND, 2017

KARL SCHWEDE

0.1. **F-splitting's implications for local cohomology continued.** Last time we proved the following.

Corollary 0.1. *Suppose that Frobenius acts injective only $H^i_I(R)$ for some $I \subseteq R$ and $i > 0$. Further suppose that $J \cdot H^i_I(R) = 0$, then also $\sqrt{J} \cdot H^i_I(R) = 0$. In particular, in an F-injective local ring $(R, \mathfrak{m}, k)$, if $H^i_{\mathfrak{m}}(R)$ has finite length then $H^i_{\mathfrak{m}}(R)$ is a k-vector space.*

There are ways to weaken the Cohen-Macaulay condition which appear in the commutative algebra literature. We give only the definitions that are convenient for our purposes.

Definition 0.2. A Noetherian d-dimensional local ring $(R, \mathfrak{m}, k)$ is called *quasi-Buchsbaum* if each $H^i_{\mathfrak{m}}(R)$ is a finite dimension k-vector space for each $i < \dim R$. It is called *quasi-Buchsbaum* if when we consider an exact triangle

$$K^{\bullet} \longrightarrow \mathbf{R}\Gamma_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(R)[-d] \xrightarrow{+1}$$

then $K^{\bullet} \in D^b(k)$.[1] In particular, since every short exact sequence of K vector spaces is split, $K^{\bullet}$ is quasi-isomorphic to the direct sum of its cohomologies (appropriately shifted).

It was known that for F-pure rings, being quasi-Buchsbaum implies the Buchsbaum condition [GO83], but it had been an open question popularized by S. Takagi whether this implication also holds for F-injective rings. This was shown recently to be the case by L. Ma [Ma15]. We give a proof of this now due to B. Bhatt, L. Ma and the author which can be found in [BMS16]. We first need a lemma.

Proposition 0.3. *Let $A \longrightarrow B$ be a surjection of Noetherian rings with induced surjection $A^{\infty} \longrightarrow B^{\infty}$. Let $K^{\bullet} \in D^b(A^{\infty})$ be a complex such that each $h^i(K^{\bullet})$ is a B^{∞}-module. Then $K^{\bullet} \simeq K^{\bullet} \otimes^{\mathbf{L}}_{A^{\infty}} B^{\infty}$ via the canonical map, and thus $K^{\bullet}$ comes from $D^b(B^{\infty})$ via the forgetful functor $D^b(B^{\infty}) \longrightarrow D^b(A^{\infty})$.*

Proof. We must check that the canonical map $K^{\bullet} \longrightarrow K^{\bullet} \otimes^{\mathbf{L}}_{A^{\infty}} B$ is an isomorphism for $K^{\bullet}$ as above. We first prove the result for $K = M[0]$ being a B-module M placed in degree 0. But then $K^{\bullet} \otimes^{\mathbf{L}}_{A^{\infty}} B^{\infty} \simeq M[0] \otimes^{\mathbf{L}}_{B^{\infty}} (B^{\infty} \otimes^{\mathbf{L}}_{A^{\infty}} B^{\infty})$, so the claim follows from **??**.

For the general case, we induct on the maximum length $l = j - i$ such that $h^j(K^{\bullet}) \neq 0$ and $h^i(K^{\bullet}) \neq 0$. We have already handled the base case since a complex that has cohomologies only in degree zero is quasi-isomorphic to a module viewed as a complex in that degree. Next suppose that the result is true in the case of $\leq l$ and consider a complex $K^{\bullet}$ where $h^j(K^{\bullet}) \neq 0$ and $h^i(K^{\bullet}) \neq 0$ where $l + 1 = j - i$.

[1]Note $H^d_{\mathfrak{m}}(R)$ is the top cohomology of $\mathbf{R}\Gamma_{\mathfrak{m}}(R)$ and so there always exists such a map.

Consider the exact triangle

$$h^i(K^\bullet)[-i] \to K^\bullet \to C^\bullet \xrightarrow{+1}$$

$C^\bullet$ is just the truncation of $K^\bullet$ at the ith spot and so the inductive hypothesis implies that $C^\bullet \simeq_{\text{qis}} C^\bullet \otimes^{\mathbf{L}}_{A^\infty} B^\infty$. Thus we have

$$\begin{array}{ccccccc} h^i(K^\bullet)[-i] & \longrightarrow & K^\bullet & \longrightarrow & C^\bullet & \xrightarrow{+1} & \\ \downarrow \sim & & \downarrow & & \downarrow \sim & & \\ h^i(K^\bullet)[-i] \otimes^{\mathbf{L}}_{A^\infty} B^\infty & \longrightarrow & K^\bullet \otimes^{\mathbf{L}}_{A^\infty} B^\infty & \longrightarrow & C^\bullet \otimes^{\mathbf{L}}_{A^\infty} B^\infty & \xrightarrow{+1} & \end{array}$$

The the vertical maps on the ends are quasi-isomorphisms and thus so is the map in the middle which proves the proposition. ☐

Theorem 0.4. *Let $(R, \mathfrak{m}, k)$ be a local d-dimensional F-injective ring of characteristic $p > 0$ such that $H^i_{\mathfrak{m}}(R)$ has finite length for $i < d$. Fix $K^\bullet \in D(R)$ as the $< d$-truncation of $\mathbf{R}\Gamma_{\mathfrak{m}}(R)$. Then $K^\bullet \in D^b(k)$ and hence R is Buchsbaum.*

Proof. We prove this only in the case when k is perfect (for simplicity) where we see that $R^\infty/\mathfrak{m}^\infty = k$. For the general case (with a proof along the same lines), see [BMS16].

Note $K^\bullet$ still has a Frobenius map $F : K^\bullet \to K^\bullet$ which is injective on its cohomologies (which are the $H^i_{\mathfrak{m}}(R)$. But since the cohomologies have finite length, and so are finite dimensional k-vector spaces by Corollary 0.1, the Frobenius map is bijective on the cohomologies of $K^\bullet$.

Define $K^\bullet_\infty$ to be the $< d$-truncation of $\mathbf{R}\Gamma_{\mathfrak{m}}(R^\infty)$ and note that

$$h^i(K^\bullet_\infty) = \varinjlim_e F^e_* h^i(K^\bullet) = h^i(K^\bullet)$$

where the second to last equality follows from the fact that Frobenius acts bijectively on the cohomology of $h^i(K^\bullet)$. It follows that the canonical map $K^\bullet \to K^\bullet_\infty$ is a quasi-isomorphism. But now $K^\bullet \simeq_{\text{qis}} K^\bullet_\infty \simeq_{\text{qis}} K^\bullet_\infty \otimes_{R^\infty} k$, which shows that $K^\bullet$ is quasi-isomorphic to a complex of k-vector spaces, as claimed. ☐

0.2. **Serre's conditions and Hartog's Phenomenon.** Suppose that S has dimension n and depth m. If $\mathfrak{q}$ is an ideal of height say $n-1$, it is possible that $\operatorname{depth}_{S_\mathfrak{q}} S_\mathfrak{q} = m$ and it is *also* possible that $\operatorname{depth}_{S_\mathfrak{q}} S_\mathfrak{q}$ has depth $m-1$. For example, consider R from **??** and set $S = R[w]$. It is not difficult to see that S has depth 2 at the origin (since if you mod out by w you get back R, which has depth 1). However, if one localizes at the prime ideal $\langle x, y, u, v\rangle$, you obtain a ring of depth 1 (you essentially get R with enlarged base field k to $k(w)$). On the other hand, localizing at $\langle x, y, u, w\rangle$ inverts v and so kills x and y and thus produces $k(v)[u, w]_{u,w}$, a ring of depth 2.

Because of this unpredictable behavior of depth under localization, we have the following definition.

Definition 0.5. A finitely generated module M over a Noetherian ring R is said to satisfy $\mathbf{S}_n$ if for every prime $\mathfrak{q} \in \operatorname{Spec} R$, $\operatorname{depth} M_\mathfrak{q} \geq \min\{n, \dim R_\mathfrak{q}\}$.[2]

In particular, an $\mathbf{S}_n$-module is Cohen-Macaulay in codimension n and has depth $\geq n$ elsewhere.

[2]In some published work, $\dim R_\mathfrak{q}$ is replaced by $\dim M_\mathfrak{q}$ in this definition.

References

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