NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 24TH, 2017

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0.1. Serre's conditions and Hartog's Phenomenon continued.

Definition 0.1. A finitely generated module M over a Noetherian ring R is said to satisfy $\mathbf{S}_n$ if for every prime $\mathfrak{q} \in \operatorname{Spec} R$, $\operatorname{depth} M_{\mathfrak{q}} \geq \min\{n, \dim R_{\mathfrak{q}}\}$.[1]

In particular, an $\mathbf{S}_n$-module is Cohen-Macaulay in codimension n and has depth $\geq n$ elsewhere.

A crucially important condition is $\mathbf{S}_2$, because it implies a Hartog's-like phenomenon. Before we do that, let's make a simple observation.

Lemma 0.2. *If $(R, \mathfrak{m})$ is a local ring, M is a module of depth ≥ 2, and $U = \operatorname{Spec} \setminus \mathfrak{m}$, then*

$$M \cong \Gamma(U, M).$$

Proof. We have an exact sequence $H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow \Gamma(U, M) \longrightarrow H^1_{\mathfrak{m}}(M)$ and the two local cohomologies are zero by the depth condition. $\square$

The point is that if a module is has depth ≥ 2, then it is completely determined by its behavior outside the origin. A more general statement holds when the module is $\mathbf{S}_2$.

Theorem 0.3. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of dimension ≥ 2 and that M is an $\mathbf{S}_2$-module. If $I \subseteq R$ is an ideal such that $\dim V(I) \leq \dim R - 2$ and $U = \operatorname{Spec} R \setminus V(I)$, then $M \cong \Gamma(U, M)$.*

Proof. We need to show that $M \longrightarrow \Gamma(U, M)$ is bijective and so let K be the kernel of C be the cokernel. Let Q be a minimal prime in the support of $K \oplus C$. In particular, $M_Q \longrightarrow \Gamma(U, M)_Q = \Gamma(U \cap \operatorname{Spec} R_Q, M_Q)$ is not bijective and $(K \oplus C)_Q$ is supported only at the maximal ideal. Since $\Gamma(U, M)_Q = M_Q$ (essentially by definition) if Q has height 1, we may assume that Q has height at least 2. Thus $\operatorname{depth} M_Q \geq 2$ and so by Lemma 0.2, $M_Q \longrightarrow \Gamma(U \cap \operatorname{Spec} R_Q, M_Q)$ is bijective, a contradiction. $\square$

Remark 0.4. $\mathbf{S}_2$-modules are often viewed as the modules that are determined by their behavior in codimension 1.

1. Local duality and Gorenstein rings

In this section we state local duality. First we need a brief review of injective hulls.

Definition 1.1. Suppose that R is a ring and M is an R-module. An overmodule $E \supseteq M$ is said to be an *essential extension of* M if for every submodule $D \subseteq E$, if $D \cap M = 0$ then $D = 0$.

An *injective hull* $E(M)$ of M is an essential extension of M that is also injective as an R-module.

[1] In some published work, $\dim R_{\mathfrak{q}}$ is replaced by $\dim M_{\mathfrak{q}}$ in this definition.

Fact 1.2.
- ∘ Injective hulls exist for any module M.
- ∘ Injective hulls are unique up to *non*-unique isomorphism (fixing M).
- ∘ The formation of injective hulls commutes with localization, $W^{-1}E_R(M) = E_{W^{-1}R}(W^{-1}M)$.

Notation 1.3. For the rest of the semester, if $(R, \mathfrak{m}, k)$ is a local ring, then $E = E_{R/\mathfrak{m}} = E_k$ will denote the injective hull of $k = R/\mathfrak{m}$.

Example 1.4. In the case that $(R, \mathfrak{m}, k)$ is a regular local Noetherian ring (or more generally a Gorenstein[2] local Noetherian ring), $E_k \cong H^{\dim R}_{\mathfrak{m}}(R)$. This will follow from local duality below, but that's not the right way to prove it.

Let's quickly state Matlis duality which roughly says that Homing into the injective hull of the residue field of a local ring does not kill (much) information.

Theorem 1.5 (Matlis Duality)**.** *Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring. Then:*

1. *The functor $T(\_\_) = \operatorname{Hom}_R(\_\_, E)$ is faithful on the category of finitely generated R-modules and also on the category of Artinian R-modules.*
2. *For Artinian modules N, $T(T(N)) \cong N$.*
3. *For Noetherian modules M, $T(T(M)) \cong \widehat{M} = M \otimes_R \widehat{R}$.*
4. *$T(\_\_)$ takes modules of finite length to modules of the same finite length.*

If in addition R is complete then

5. *$T(\_\_)$ induces an antiequivalence[3] of categories*
$$\left\{ \begin{array}{c} \textit{Noetherian} \\ R\textit{-modules} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Artinian} \\ R\textit{-modules} \end{array} \right\}.$$

We now define dualizing complexes.

Definition 1.6. Suppose that R is a Noetherian ring. An object $\omega^{\bullet} \in D^b_{f.g.}(R)$ is called a *dualizing complex for R* if the following two conditions are satisfied.

- (a) $\omega^{\bullet}$ has finite injective dimension (is quasi-isomorphic to a bounded complex of injectives) and,
- (b) The functor $\mathbf{D}(\_\_) = \mathbf{R}\operatorname{Hom}_R(\_\_, \omega^{\bullet})$ has the property that the canonical map $C^{\bullet} \to \mathbf{D}(\mathbf{D}(C^{\bullet}))$ is an isomorphism for all $C^{\bullet} \in D_{f.g.}(R)$.
- (b') Or equivalently to (b), $R \cong \mathbf{R}\operatorname{Hom}_R(\omega^{\bullet}, \omega^{\bullet})$.

Exercise 1.1. Prove that (b') implies (b) above.

Fact 1.7. Dualizing complexes are unique up to two operations.

- ∘ Shift (if $\omega^{\bullet}$ is a dualizing complex, so is $\omega^{\bullet}[n]$).
- ∘ Tensoring with rank-1 projective modules (if P is a projective module of rank-1[4] then if $\omega^{\bullet}$ is a dualizing complex, so is $\omega^{\bullet}[n]$).

Lemma 1.8. *If $\omega^{\bullet}$ is a dualizing complex for a Noetherian ring R and W is a multiplicatively closed set, then $W^{-1}\omega^{\bullet}$ is a dualizing complex for $W^{-1}R$.*

Proof. Inverting multiplicatively closed sets preserves injectives in Noetherian rings and so condition (a) is fine in the definition of the dualizing complex. The same operation also preserves the isomorphism of (b'). □

[2]to be defined soon
[3]*i.e.* , a contravariant equivalence
[4]Meaning that $P_Q \cong R_Q$ for all $Q \in \operatorname{Spec} R$.

Definition 1.9. A ring R is called *Gorenstein* if R has finite injective dimension as an R-module. Note for a Gorenstein ring, R is its own dualizing complex.

Lemma 1.10. *A Noetherian ring R is Gorenstein if and only if R has a dualizing complex $\omega_R^{\bullet} \cong M[n]$ such that M is projective of rank 1.*

Example 1.11. Suppose that R is a regular local ring, then R is Gorenstein. To see this let $d = \dim R$ and observe that

$$\operatorname{Ext}^{d+1}(M, R) = 0$$

for all R-modules M (because R has finite global dimension d). But since this holds for all R-modules M, it implies that R itself has finite injective dimension.

More generally, if R is a regular ring it is also Gorenstein.

Remark 1.12. Not every ring has a dualizing complex, but nearly all the rings we care about do. In particular, any ring that is a quotient of a regular ring (or more generally a Gorenstein ring) has a dualizing complex. In particular, if $R = S/I$ where S is regular, then

$$\omega_R^{\bullet} = \mathbf{R}\operatorname{Hom}_S(R, S)$$

is a dualizing complex. We'll prove this next time.

References