NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 27TH, 2017

KARL SCHWEDE

Remark 0.1. Not every ring has a dualizing complex, but nearly all the rings we care about do. In particular, any ring that is a quotient of a regular ring (or more generally a Gorenstein ring) has a dualizing complex. In particular, if $R = S/I$ where S is regular, then

$$\omega_R^\bullet = \mathbf{R}\operatorname{Hom}_S(R, S)$$

is a dualizing complex. It is certainly of finite injective dimension because if I is an injective S-module, $\operatorname{Hom}_S(R, I)$ is an injective R-module (so if we take a finite injective resolution of S applying $\operatorname{Hom}_S(R, \_\_)$ gives a complex of R-modules finite injective dimension) and then observe that

$$\begin{array}{rl} & \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_S(M, S), \mathbf{R}\operatorname{Hom}_S(R, S)) \\ \cong & \mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_S(M, S) \otimes_R^L R, S) \\ \cong & \mathbf{R}\operatorname{Hom}_S(\mathbf{R}\operatorname{Hom}_S(M, S), S) \\ \cong & M \end{array}$$

where the last $\cong$ holds because S is a dualizing complex for S. In fact the argument we just used implies that if $S \longrightarrow R$ is any map of rings such that R is a finite S-module, and if $\omega_S^\bullet$ is a dualizing complex for S then $\mathbf{R}\operatorname{Hom}_S(R, \omega_S^\bullet)$ is a dualizing complex for R.

Corollary 0.2. *Suppose that $(R, \mathfrak{m})$ is an F-finite Noetherian local ring of characteristic $p > 0$ with a dualizing complex. Then $\operatorname{Hom}_R(F^e_* R, \omega_R^\bullet) =: \omega^\bullet_{F^e_* R}$ is a dualizing complex for $F^e_* R$. Furthermore, $F^e_* \omega_R^\bullet \simeq_{qis} \omega^\bullet_{F^e_* R}$.*

Exercise 0.1. Prove Corollary 0.2.

Hint: The fact that it is a dualizing complex is just what was worked out in the above example. Dualizing complexes are unique up to shifting and twisting by rank-one projectives. Use the fact that it is local to handle the rank-1 projective case. Then use the that the localization of a dualizing complex is a dualizing complex to handle the shift (localize at a minimal prime).

Remark 0.3. With the notation in the above corollary, the induced map

$$F^e_* \omega_R^\bullet \longrightarrow \omega_R^\bullet$$

dual to $R \longrightarrow F^e_* R$ is often called the trace of Frobenius for reasons that we will see later.

Example 0.4. If R is Gorenstein with dualizing complex $\omega_R^\bullet \simeq_{\text{qis}} \omega_R[n]$ and if $f \in R$ is a nonzero divisor, then R/f is Gorenstein with dualizing complex

$$\omega^\bullet_{R/f} = \omega_R/f[n-1] = \operatorname{Ext}^1(R/f, \omega_R[n])$$

To see this, consider the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot f} R \longrightarrow R/f \longrightarrow 0$$

and apply $\mathbf{R}\operatorname{Hom}_R(\_\_, \omega_R^{\bullet}) = \mathbf{R}\operatorname{Hom}_R(\_\_, \omega_R[n])$. We obtain

$$\mathbf{R}\operatorname{Hom}_R(R/f, \omega_R[n]) \longrightarrow \mathbf{R}\operatorname{Hom}_R(R, \omega_R[n]) \xrightarrow{\cdot f} \mathbf{R}\operatorname{Hom}_R(R, \omega_R[n]) \xrightarrow{+1}$$

and taking cohomology (starting at $-n$) yields

$$0 \longrightarrow \omega_R \xrightarrow{\cdot f} \omega_R \longrightarrow \operatorname{Ext}^1(R/f, \omega_R) \longrightarrow 0 \longrightarrow \dots$$

where the first zero $0 = \operatorname{Hom}_R(R/f, \omega_R)$ holds because ω_R is locally free and in particular, torsion free.

Finally, in order to state local duality, we need one more definition.

Definition 0.5. Suppose that $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring with dualizing complex $\omega_R^{\bullet} \in D^b_{f.g}(R)$. The dualizing complex $\omega_R^{\bullet}$ is called *normalized* if $h^j(\omega_R^{\bullet}) = 0$ for $j < -d$ and $h^{-d}(\omega_R^{\bullet}) \neq 0$.

The module $h^{-d}(\omega_R^{\bullet})$ is called the *canonical module* and typically denoted by ω_R.

Remark 0.6. A normalized dualizing complex does not remain normalized after localization. In particular, if $\omega_R^{\bullet}$ is normalized for $(R, \mathfrak{m})$ and $\mathfrak{q} \in \operatorname{Spec} R$ is a prime, then $(\omega_R^{\bullet})_{\mathfrak{q}}$ is not normalized even though it is still surjective.

We now state local duality.

Theorem 0.7 (Local duality). *Suppose that $(R, \mathfrak{m}, k)$ is a local Noetherian domain with dualizing complex $\omega_R^{\bullet}$ and $C^{\bullet} \in D^b_{f.g.}(R)$ and injective hull of the residue field E. Then*

$$\operatorname{Hom}(\mathbf{R}\operatorname{Hom}_R(C^{\bullet}, \omega_R^{\bullet}), E) \simeq_{qis} \mathbf{R}\Gamma_{\mathfrak{m}}(C^{\bullet}).$$

In the case that R is complete, this can also be written as

$$\mathbf{R}\operatorname{Hom}_R(C^{\bullet}, \omega_R^{\bullet}) \simeq_{qis} \operatorname{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}}(C^{\bullet}), E).$$

Note we do not need to derive the $\operatorname{Hom}(\_\_, E)$ functor since E is injective.

We now specialize this to other settings in order to deduce standard facts (this is not the right way to prove these facts, but it's not a bad way to remember their statements).

Corollary 0.8. *A Noetherian local ring $(R, \mathfrak{m})$ with a dualzing complex $\omega_R^{\bullet}$ is Cohen-Macaulay if and only if the dualizing complex for R is centered in one degree,*

$$\omega_R^{\bullet} \simeq_{qis} \omega_R[n].$$

Corollary 0.9. *With notation as above, if $\omega_R^{\bullet}$ is a normalized dualizing complex then $\operatorname{Hom}_R(\omega_R^{\bullet}), E) \cong \mathbf{R}\Gamma_{\mathfrak{m}}(R)$.*

While the above is obvious from the definition.

Corollary 0.10. *With notation as above, R is Gorenstein if and only if the normalized dualizing complex $\omega_R^{\bullet} \simeq_{qis} R[\dim R]$.*

Definition 0.11. A local ring $(R, \mathfrak{m})$ with a dualizing complex is called *quasi-Gorenstein* (or 1-Gorenstein) if $\omega_R \cong R$.

Corollary 0.12. *An F-injective quasi-Gorenstein F-finite local ring is F-split.*

Proof. We'll prove this next time. □

References