NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 3RD, 2017

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Recall from last time that we proved:

Lemma 0.1. [BS15, Lemma 3.16, Lemma 5.10] *If $R \xleftarrow{g} S \xrightarrow{h} R'$ is are surjections of Noetherian rings of characteristic $p > 0$ with induced surjection $R^\infty \xleftarrow{g^\infty} S^\infty \xrightarrow{h^\infty} R'^\infty$ of perfect rings. Then $\mathrm{Tor}^i_{S^\infty}(R^\infty, R'^\infty) = 0$ for all $i \neq 0$ or in other words*

$$R^\infty \otimes^{\mathbf{L}}_{S^\infty} R'^\infty \simeq_{qis} R^\infty \otimes_{S^\infty} R'^\infty.$$

In particular, specializing to the case $R' = R$, the multiplication map $R^\infty \otimes^{\mathbf{L}}_{S^\infty} R^\infty \longrightarrow R^\infty$ is a quasi-isomorphism.

Remark 0.2. In [BS15], they obtained a more general statement. They started with perfect rings (that were not necessarily the perfections of Noetherian rings).

Proposition 0.3. [BS15, Proposition 5.31] *Let R^∞ be the perfection of a complete local ring R. Then R^∞ has finite global dimension.*

Proof. Write $R = S/I$ for $S = k[\![x_1, \dots, x_n]\!]$ and note we still have a surjection $S^\infty \longrightarrow R^\infty$.

Let M be an arbitrary R^∞-module. Then

$$M = M \otimes^{\mathbf{L}}_{R^\infty} R^\infty = M \otimes^{\mathbf{L}}_{R^\infty} (R^\infty \otimes^{\mathbf{L}}_{S^\infty} R^\infty) = M \otimes^{\mathbf{L}}_{S^\infty} R^\infty$$

by Lemma 0.1. We are trying to show that $\mathbf{R}\mathrm{Hom}_R(M, N)$ has (uniformly) bounded cohomology but

$$\begin{array}{rl} & \mathbf{R}\mathrm{Hom}_{R^\infty}(M, N) \\ = & \mathbf{R}\mathrm{Hom}_{R^\infty}(M \otimes^{\mathbf{L}}_{S^\infty} R^\infty, N) \\ = & \mathbf{R}\mathrm{Hom}_{S^\infty}(M, \mathbf{R}\mathrm{Hom}_{R^\infty}(R^\infty, N)) \\ = & \mathbf{R}\mathrm{Hom}_{S^\infty}(M, N). \end{array}$$

and so it suffices to prove the result of S^∞. Let $d = \dim S$ and note that $S^\infty = \lim_{\longrightarrow} S^{1/p^e}$. Now, we can view M as an S^{1/p^e}-module by restriction, and write $F_e = M \otimes_{S^{1/p^e}} S^\infty$ for its base change back to S^∞. Obviously we have maps

$$\dots \longrightarrow M \otimes_{S^{1/p^e}} S^\infty \longrightarrow M \otimes_{S^{1/p^{e+1}}} S^\infty \longrightarrow \dots \longrightarrow M$$

The direct limit $\lim_{\longrightarrow} F_e$ is in fact equal to M since any element of S^∞ is in some S^{1/p^e} for $e \gg 0$. On the other hand, M has projective dimension $\leq d$ as an S^{1/p^e}-module and so F_e has projective dimension $\leq d$ as an S^∞-module.

Now consider

$$\bigoplus F_e \to \bigoplus F_e$$

where $(\dots, 0, a_e, 0, 0 \dots) \mapsto (\dots, 0, a_e, -a_e, 0, \dots)$ and the remainder of the map is defined by linearity. The cokernel of this map is exactly $\lim_{\longrightarrow} F_e = M$ and the map is clearly injective. Hence the projective dimension of M is $\leq d + 1$ which completes the result. $\square$

Corollary 0.4. *Suppose that R is a complete Noetherian local domain of characteristic $p > 0$ and that R^∞ is a flat R-module, then R is regular.*

Proof. By Proposition 0.3, every R^∞-module has finite global-dimension. On the other hand, $R \longrightarrow R^\infty$ is also *faithfully* flat by hypothesis.

We now prove that R has finite global dimension. Let n denote the global-dimension of R^∞. Suppose that M, N are R-modules with M finite such that $\mathrm{Ext}^i_R(M, N) \neq 0$ for some $i > n$ (recall you can verify global dimension just for finitely generated modules). Then $\mathrm{Ext}^i_R(M, N) \otimes_R R^\infty \neq 0$ by the faithfulness of R^∞/R. But $\mathrm{Ext}^i_R(M, N) \otimes_R R^\infty = \mathrm{Ext}^i_{R^\infty}(M \otimes_R R^\infty, N \otimes_R R^\infty)$ by the flatness of R^∞/R and the fact that M is finitely presented. The fact that this is nonzero contradicts n being the global-dimension of R^∞ and proves the claim.

Now that R has finite global dimension, we conclude that R is regular as claimed. □

1. Frobenius splittings

We saw that Frobenius is flat if and only if the ring is regular. It is then natural to ask, how can we weaken the condition that F_*R is flat. We are primarily interested in the case that F_*R is a finite and hence locally free R-module, thus consider the following.

Proposition 1.1. *Suppose $R^{1/p} \cong R \oplus M$ as R-module, then there exists a Frobenius splitting of $R \longrightarrow F_*R$. In particular regular local rings are Frobenius split.*

Proof. There exists a surjective R-linear map $\phi : F_*R \longrightarrow R$ (project onto the factor). Say $\phi(F_*a) = 1$. Consider the new R-linear map

$$\psi(F_*\_\_) = \phi(F_*(a \cdot \_\_)),$$

it splits Frobenius since it sends F_*1 to 1. □

References

[BS15] B. Bhatt and P. Scholze: *Projectivity of the Witt vector affine Grassmannian*, arXiv:1507.06490.