NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 6TH, 2017

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We saw that Frobenius is flat if and only if the ring is regular. It is then natural to ask, how can we weaken the condition that F_*R is flat. We are primarily interested in the case that F_*R is a finite and hence locally free R-module, thus consider the following.

Proposition 0.1. *Suppose R is a regular Noetherian ring of characteristic $p > 0$ and that F_*R is a locally free finite R-module. Then there exists an R-linear map $F_*R \longrightarrow R$ sending $F_*1 \mapsto 1$, a* Frobenius splitting.

Proof. First suppose that R is a regular local ring. Then F_*R being locally free implies that F_*R is actually free as an R-module. In particular, there exists a surjective R-linear map $\phi : F_*R \longrightarrow R$ (project onto one of the factors). Say $\phi(F_*a) = 1$. Consider the new R-linear map

$$\psi(F_*\_\_) = \phi(F_*(a \cdot \_\_)).$$

It satisfies $\psi(F_*1) = 1$ and so we have handled the case when R is local.

Now suppose that R is not local, consider the map $\sigma : \mathrm{Hom}_R(F_*R, R) \longrightarrow R$ which sends $\phi \mapsto \phi(F_*1)$. It is easy to see that this is a map of R-modules hence it is surjective if and only if it is locally surjective. On the other hand if σ is surjective, the existence of the desired map is produced. Hence it suffices to show that

$$\mathrm{Hom}_R(F_*R, R)_{\mathfrak{m}} \longrightarrow R_{\mathfrak{m}}$$

is surjective (where $\mathfrak{m}$ is some maximal ideal of R). On the other hand

$$\mathrm{Hom}_R(F_*R, R)_{\mathfrak{m}} \cong \mathrm{Hom}_{R_{\mathfrak{m}}}((F_*R)_{\mathfrak{m}}, R_{\mathfrak{m}}) \cong \mathrm{Hom}_{R_{\mathfrak{m}}}(F_*R_{\mathfrak{m}}, R_{\mathfrak{m}})$$

and it is not difficult to see that our localized map above agrees (via this isomorphism) with the evaluation at F_*1 map $\mathrm{Hom}_{R_{\mathfrak{m}}}(F_*R_{\mathfrak{m}}, R_{\mathfrak{m}}) \longrightarrow R_{\mathfrak{m}}$. This completes the proof. $\square$

Thus it is clear that having a is a potential weakening of being regular (in fact, it is quite close to the notion of being semi-log canonical from birational algebraic geometry). This leads us to our next section.

1. Frobenius split rings

Definition 1.1. A ring R containing a field of characteristic $p > 0$ is called *(locally) Frobenius split* if there exists an R-linear map $\phi : F_*R \longrightarrow R$ such that $\phi(F_*1) = 1$. The map ϕ is called a *Frobenius splitting*.

Frobenius splittings behave best when F_*R is a finitely generated (thus finitely presented in the Noetherian) R-module. Because of this, we make the following definition.

Definition 1.2. A ring R of characteristic $p > 0$ is said to be *F-finite* if F_*R is a finite R-module.

We will see later that F-finite rings avoid most of the pathologies that other arbitrary rings can satisfy.

Lemma 1.3. *The following are equivalent.*

(a) R is F-split.
(b) The map $R \longrightarrow F^e_* R$ is split for some $e > 0$.
(c) The map $R \longrightarrow F^e_* R$ is split for all $e > 0$.
(d) There exists a surjective R-linear map $F^e_* R \longrightarrow R$ for all $e > 0$.
(e) There exists a surjective R-linear map $F^e_* R \longrightarrow R$ for some $e > 0$.

Proof. (a) $\Rightarrow$ (b) follows simply from taking $e = 1$. We now show (b) $\Rightarrow$ (c). First if $R \longrightarrow F^e_* R$ splits, then so does $R \longrightarrow F_* R \longrightarrow F^e_* R$, and hence so does $R \longrightarrow F_* R$. Thus any $e > 1$ implies that $e = 1$ case so let $\phi : F_* R \longrightarrow R$ be the map which sends $F_* 1$ to 1. Then $\phi \circ (F_* \phi) : F^2_* R \longrightarrow R$ sends $F^2_* 1$ to 1 as well. Likewise $\phi \circ (F_* \phi) \circ \cdots \circ (F^{n-1}_* \phi) : F^n_* R \longrightarrow R$ sends $F^n_* 1 \mapsto 1$ as desired.

Next, obviously (c) $\Rightarrow$ (d) since if 1 is contained in the image, then so are all multiplies of 1 (the entire ring). Furthermore (d) $\Rightarrow$ (e) and so it suffices to show that (e) $\Rightarrow$ (a). Let $\phi : F^e_* R \longrightarrow R$ be a surjective map with $\phi(F^e_* a) = 1$. Then forming $\psi(F^e_* \_\_) = \phi(F^e_*(a \cdot \_\_))$ shows that there exists a splitting $\psi : F^e_* R \longrightarrow R$. Forming the composition $F_* R \longrightarrow F^e_* R \xrightarrow{\psi} R$ constructs a splitting of Frobenius. ☐

Lemma 1.4. *A Frobenius split ring is reduced.*

Proof. Suppose R is not reduced but it is Frobenius split. Then there exists some $0 \neq r \in R$ with $r^p = 0$. Let $\phi : F_* R \longrightarrow R$ be a Frobenius splitting then we have the composition:

$$
\begin{array}{ccccc}
R & \xrightarrow{F} & F_* R & \xrightarrow{\phi} & R \\
r & \longmapsto & F_* r^p & \longmapsto & r.
\end{array}
$$

But the middle term is zero, a contradiction. ☐

Exercise 1.1. Suppose R is an F-finite Noetherian ring. Show that R is F-split if and only if $R_{\mathfrak{m}}$ is F-split for every maximal ideal $\mathfrak{m} \subseteq R$.

At this point, we don't even know that *any* interesting examples of F-split rings that are not regular. There's a good way to construct lots of them however.

Theorem 1.5. *Suppose that $R \subseteq S$ is an extension of rings such that there exists a surjective R-linear map $T : S \longrightarrow R$. Then if S is F-split, so is R.*

Proof. Via the argument we used in Lemma 1.3, we may assume that the map $T : S \longrightarrow R$ sends $1_S \mapsto 1_R$. Let $\phi_S : F_* S \longrightarrow S$ be a Frobenius splitting. We have the following composition:

$$F_* R \hookrightarrow F_* S \xrightarrow{\phi_S} S \xrightarrow{T} R.$$

It is R-linear and it is easy to check that it sends $F_* 1_R \longrightarrow 1_R$. Thus R is F-split. ☐

Example 1.6. Consider $R = k[x^2, xy, y^2] \subseteq k[x, y] = S$ where k is an F-finite field of characteristic $p > 0$. Obviously S is F-split since it is regular. On the other hand

$$S = k[x, y] = k \bigoplus (k \cdot x \oplus k \cdot y) \bigoplus (k \cdot x^2 \oplus k \cdot xy \oplus k \cdot y^2) \bigoplus \ldots.$$

R is just the subring of even degree terms, and hence it is clear that $R \subseteq S$ splits.

Definition 1.7. Suppose that $R \subseteq S$ is an extension of rings. If there exists a splitting $S \longrightarrow R$ (or equivalently any surjective map $S \longrightarrow R$) then we say that the extension $R \subseteq S$ *splits* and that *R is a summand of S.*

Just as we did in the example:

Corollary 1.8. *If R is an F-finite Noetherian ring of characteristic $p > 0$ that is a summand of a regular ring, then R is F-split.*

It turns out that summands of regular rings are really quite common!

2. Fedder's criterion and computations

The main goal is to prove Fedder's Lemma, **??**, a remarkably useful tool for explicitly working with p^{-e}-linear maps (equivalently R-linear maps $F^e_* R \longrightarrow R$). For instance, using Fedder's Lemma it is easy to determine whether a given F-finite ring is F-split. The organization of this section is as follows. First we prove Fedder's lemma and some corollaries, we then do numerous computations with Fedder's lemma. Finally, we discuss Fedder's criterion outside the F-finite case and define F-purity in general.

2.1. **Fedder's Lemma on p^{-e}-linear maps.** We begin with the following:

Notation 2.1. Throughout the rest of the section, k is an F-finite field and $S = k[x_1, \ldots, x_n]$, or a localization thereof, or $S = k[\![x_1, \ldots, x_n]\!]$.

Some of the facts below we have proven previously, but we recall them for ease of reference.

Lemma 2.2. *Suppose that k is an F-finite field, $S = k[x_1, \ldots, x_n]$, (or its localization at the origin, or $S = k[\![x_1, \ldots, x_n]\!]$). Then $\mathrm{Hom}_S(F^e_* S, S)$ is isomorphic to $F^e_* S$ as an $F^e_* S$-module with generator equal to the following map:*

$$\Phi_S(F^e_* \mathbf{x}^\lambda) = \left\{ \begin{array}{ll} 1 & \text{if } \lambda_1 = \cdots = \lambda_n = p^e - 1 \\ 0 & \text{otherwise} \end{array} \right\}$$

defined on the a basis $\{a_i \mathbf{x}^\lambda\}$ where the a_i form a basis for $F^e_* k$ over k, $a_1 = 1$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfies $0 \leq \lambda_i \leq p^e - 1$.

References