NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA FEBRUARY 8TH, 2017

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0.1. **Fedder's Lemma on p^{-e}-linear maps.** We begin with the following:

Notation 0.1. Throughout the rest of the section, k is an F-finite field and $S = k[x_1, \ldots, x_n]$, or a localization thereof, or $S = k[\![x_1, \ldots, x_n]\!]$.

Some of the facts below we have proven previously, but we recall them for ease of reference.

Example 0.2. Consider the polynomial ring $S = k[x_1, \ldots, x_n]$ for k an F-finite field. Then S is also F-finite and of course $F^e_* S$ is a free S-module with basis $\{F_* a_i \mathbf{x}^\lambda\}$ where the $F^e_* a_i$ are a basis for $F^e_* k$ over k and $\mathbf{x}^\lambda$ denotes the monomials of the form $x_1^{\lambda_1} \ldots x_n^{\lambda_n}$ such that $0 \leq \lambda \leq p^e - 1$.

Lemma 0.3. *Suppose that k is an F-finite finite field, then* $\mathrm{Hom}_k(F^e_* k, k) \cong F_* k$ *as $F_* k$-modules.*

Proof. Suppose that $m = [F^e_* k : k]$. Obviously $\mathrm{Hom}_k(F^e_* k, k)$ has rank m as a k-module since it is the dual of a rank m vector space. On the other hand, if an $F^e_* k$-module has rank m as a k-module, it clearly has rank 1 as an $F^e_* k$-module, and so the result follows. □

Lemma 0.4. *Suppose that k is an F-finite field, $S = k[x_1, \ldots, x_n]$, (or its localization at the origin, or $S = k[\![x_1, \ldots, x_n]\!]$). Then* $\mathrm{Hom}_S(F^e_* S, S)$ *is isomorphic to $F^e_* S$ as an $F^e_* S$-module with generator equal to the following map:*

$$\Phi_S(F^e_* \mathbf{x}^\lambda) = \left\{ \begin{array}{ll} 1 & \text{if } \lambda_1 = \cdots = \lambda_n = p^e - 1 \\ 0 & \text{otherwise} \end{array} \right\}$$

defined on the a basis $\{a_i \mathbf{x}^\lambda\}$ where the a_i form a basis for $F^e_* k$ over k, $a_1 = 1$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfies $0 \leq \lambda_i \leq p^e - 1$.

Proof. We first do the case where k is perfect. To see that it is cyclic, it is sufficient to show that each of the projections $\rho_{\mathbf{x}^\lambda} : F^e_* S \longrightarrow S$ onto the $F^e_* \mathbf{x}^\lambda$-summand is an $F_* S$-multiple of Φ_S with $\mathbf{x}^\lambda$ defined as in Example 0.2. But simply observe that

$$\rho_{\mathbf{x}^\lambda}(F^e_* \_\_) = \Phi_R((F^e_* \mathbf{x}^{\mathbf{p^e - 1 - \lambda}}) \cdot F^e_* \_\_).$$

On the other hand $\mathrm{Hom}_S(F^e_* S, S)$ is certainly torsion free and the lemma is proven when k is perfect and S is a polynomial ring.

Now assume that k is not perfect, choose a basis $\{F_* a_i\}_{i=1}^m$ of $F_* k$ over k with $a_1 = 1$. Note that $\{a_i \mathbf{x}^\lambda \mid \lambda_j = 0, \ldots, p^e - 1, i = 1, \ldots, m\}$ form a basis for $F^e_* S$ over S. Choose maps $\mu_i : F_* k \longrightarrow k$ which project onto the a_i. For each $m \geq i > 1$, choose $b_j \in k$ such that $\mu_1(F^e_* b_i \_\_) = \mu_i(\_\_)$. On the other hand each $F_* k \longrightarrow k$ induces

$$\nu_i : (F^e_* k)[x_1, \ldots, x_n] \longrightarrow k[x_1, \ldots, x_n]$$

by acting as the identity on the x_i. On the other hand, using the same argument we made in the perfect case, we can find maps

$$\rho_\lambda : F^e_* k[x_1, \dots, x_n] \longrightarrow (F^e_* k)[x_1, \dots, x_n]$$

projecting onto $\mathbf{x}^\lambda$ as above. Composing ρ_λ with ν_i gives us all the projections onto our basis. On the other hand, it is easy to see that those maps can all be obtained by properly pre-multiplying by appropriate $b_i \mathbf{x}^{\mathbf{p^e-1}-\lambda}$. This proves the lemma in the case of a polynomial ring. The other cases are the same. $\square$

With S as above, we next suppose that $I = \langle f_1, \dots, f_m \rangle \subseteq S$ is an ideal and set $R = S/I$. We want to relate the maps $\mathrm{Hom}_R(F^e_* R, R)$ to the maps $\mathrm{Hom}_S(F^e_* S, S)$. We begin with an easy observation in slightly greater generality.

Fix another ideal $J \subseteq S$, choose $\phi \in \mathrm{Hom}_S(F^e_* S, S)$ and pick $u \in I^{[p^e]} : J \subseteq S$. Consider the map $\psi : F^e_* S \longrightarrow S$ defined by $\psi(F^e_* z) = \phi(F^e_*(u \cdot z))$, frequently in the future we will write $\psi = (F^e_* u) \cdot \phi$ in this situation. We claim that $\psi((F^e_* J)) \subseteq I$ (this has nothing to do with S being regular). To see this claim, choose $z \in J$ and notice that $uz \in I^{[p^e]}$ can be written $uz = a_1 f_1^{p^e} + \dots + a_m f_m^{p^e}$. Then

$$\psi(F^e_* z) = \phi(F^e_*(uz)) = \phi(F^e_*(a_1 f_1^{p^e} + \dots + a_m f_m^{p^e})) = \sum_{i=1}^m \phi(F^e_* a_i) f_i \in I$$

as claimed. In fact, linearity implies that any $\psi \in \Big(F^e_*\big(I^{[p^e]} : J\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)$ satisfies $\psi(F^e_* J) \subseteq I$.

Remark 0.5. The notation $\Big(F^e_*\big(I^{[p^e]} : J\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)$ has been known to cause some confusion. The point is that $\mathrm{Hom}_S(F^e_* S, S)$ is an S-module (with S acting on either the source or target of a homomorphism) and simultaneously it is an $F^e_* S$-module, where an element $F^e_* z$ acts on $\alpha : F^e_* S \longrightarrow S$ by forming the composition

$$F^e_* S \xrightarrow{\cdot F^e_* z} F^e_* S \xrightarrow{\alpha} S.$$

Our previous observation about ψ is very useful in the case that $J = I$, consider the following diagram for any ψ satisfying $\psi(F^e_* I) \subseteq I$:

$$\begin{array}{ccc} F^e_* I & \longrightarrow & I \\ \big\downarrow & & \big\downarrow \\ F^e_* S & \xrightarrow{\psi} & S \\ \big\downarrow & & \big\downarrow \\ F^e_* R & \underset{\psi_R}{\dashrightarrow} & R \end{array}$$

In particular, each ψ determines an element of $\mathrm{Hom}_R(F^e_* R, R)$. We have just shown that:

Lemma 0.6. *There exists an $F^e_* S$-module homomorphism*

$$\rho : \Big(F^e_*\big(I^{[p^e]} : I\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S) \longrightarrow \mathrm{Hom}_R(F^e_* R, R).$$

We will study this map extensively (and in fact prove it is surjective and identify its kernel). First we make the following observations which help show what was described above.

Lemma 0.7. *With notation as above,*

(i) *Suppose that* $\phi(F^e_* J) \subseteq I$ *for all* $\phi \in \mathrm{Hom}_S(F^e_* S, S)$*, then* $J \subseteq I^{[p^e]}$.

(ii)

$$\Big(F^e_*\big(I^{[p^e]} : J\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S) = \{\psi \in \mathrm{Hom}_S(F^e_* S, S) \mid \psi(F^e_* J) \subseteq I\}.$$

Proof. With our notations for this section, $F^e_* S$ is a free S-module. Write $F^e_* S \cong S^{\oplus d}$ with basis elements $\{F^e_* g_i\}_{i=1}^d \subseteq F^e_* S$. Let $\pi_1, \ldots, \pi_d$ denote the corresponding projections. Observe that under the isomorphism $F^e_* S \cong S^{\oplus d}$ we have $F^e_* I^{[p^e]} = I \cdot F^e_* S \cong I^{\oplus d}$. Now, suppose that $\phi(F^e_* J) \subseteq I$ for all $\phi \in \mathrm{Hom}_S(F^e_* S, S)$. In particular, $\pi_i(F^e_* J) \subseteq I$ for each $\pi_1, \ldots, \pi_d$. Thus $F^e_* J$ is identified with a subset of $I^{\oplus d}$ which proves that $J \subseteq I^{[p^e]}$. This proves the first statement.

The argument before the lemma yields the containment $\subseteq$. Fix $\psi \in \mathrm{Hom}_S(F^e_* S, S)$ such that $\psi(F^e_* J) \subseteq I$. Choose a $F^e_* S$-module generator $\Phi \in \mathrm{Hom}_S(F^e_* S, S)$ by Lemma 0.4. Write $\psi = (F^e_* u) \cdot \Phi$ and observe that $\Phi(F^e_*(uJ)) \subseteq I$. Since Φ generates $\mathrm{Hom}_S(F^e_* S, S)$, we see that $uJ \subseteq I^{[p^e]}$ from part (i). Thus $u \in I^{[p^e]} : J$ and hence

$$\psi = (F^e_* u)\Phi \in \Big(F^e_*\big(I^{[p^e]} : J\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)$$

as claimed. □

Remark 0.8. The previous lemma absolutely does not hold for non-regular rings, as we shall see shortly.

Theorem 0.9 (Fedder's Lemma)**.** *With notation as above*

$$\rho : \Big(F^e_*\big(I^{[p^e]} : I\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S) \longrightarrow \mathrm{Hom}_R(F^e_* R, R)$$

is surjective and $\ker \rho$ *is isomorphic to* $\Big(F^e_* I^{[p^e]}\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)$*. In particular*

$$\mathrm{Hom}_R(F^e_* R, R) \cong \frac{\Big(F^e_*\big(I^{[p^e]} : I\big)\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)}{\Big(F^e_* I^{[p^e]}\Big) \cdot \mathrm{Hom}_S(F^e_* S, S)}.$$

References