NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 1ST, 2017

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Recall from last time.

Definition 0.1. Suppose that $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring with dualizing complex $\omega_R^{\bullet} \in D^b_{f.g}(R)$. The dualizing complex $\omega_R^{\bullet}$ is called *normalized* if $h^j(\omega_R^{\bullet}) = 0$ for $j < -d$ and $h^{-d}(\omega_R^{\bullet}) \neq 0$.

The module $h^{-d}(\omega_R^{\bullet})$ is called the *canonical module* and typically denoted by ω_R.

Theorem 0.2 (Local duality)**.** *Suppose that $(R, \mathfrak{m}, k)$ is a local Noetherian domain with dualizing complex $\omega_R^{\bullet}$ and $C^{\bullet} \in D^b_{f.g.}(R)$ and injective hull of the residue field E. Then*

$$\operatorname{Hom}(\mathbf{R}\operatorname{Hom}_R(C^{\bullet}, \omega_R^{\bullet}), E) \simeq_{qis} \mathbf{R}\Gamma_{\mathfrak{m}}(C^{\bullet}).$$

In the case that R is complete, this can also be written as

$$\mathbf{R}\operatorname{Hom}_R(C^{\bullet}, \omega_R^{\bullet}) \simeq_{qis} \operatorname{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}}(C^{\bullet}), E).$$

Corollary 0.3. *With notation as above, R is Gorenstein if and only if the normalized dualizing complex $\omega_R^{\bullet} \simeq_{qis} R[\dim R]$.*

Definition 0.4. A local ring $(R, \mathfrak{m})$ with a dualizing complex is called *quasi-Gorenstein* (or 1-Gorenstein) if the canonical module $\omega_R \cong R$.

Corollary 0.5. *An F-injective quasi-Gorenstein F-finite local ring is F-split.*

Proof. Since R is F-injective, $H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}})(F_*R)$ is injective. But this map is Matlis dual to what we called the trace $F_*\omega_R \longrightarrow \omega_R$, which now must be surjective (if it had a cokernel, then the map on local cohomology would have a cokernel, which it doesn't). But since R is quasi-Gorenstein, $\omega_R \simeq R$ and so we have a surjective map $F_*R \longrightarrow R$. This implies that R is F-split. □

We'll state some more facts about dualizing complexes (taken for example out of [Har66] or [Sta16]). For time reasons, we'll skip the proofs.

Lemma 0.6. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring with normalized dualizing complex $\omega_R^{\bullet}$ and canonical module $\omega_R = h^{-\dim R}\omega_R^{\bullet}$. Then*

- *(a) The support of ω_R is equal to the union of irreducible components of $\operatorname{Spec} R$ of maximal dimension.*
- *(b) ω_R is $\mathbf{S}_2$.*

Proof. See [Sta16, Tag 0AWE], there are some subtle points here: for example rings with dualizing complexes are catenary. □

Corollary 0.7. *Suppose that $(R, \mathfrak{m})$ is a local Noetherian domain. Then for any $0 \leq i < \dim R$, $\operatorname{Ann}_R H^i_{\mathfrak{m}}(R) \neq 0$. In particular, for each $i < H^i_{\mathfrak{m}}(R)$, there exists some $0 \neq c \in R$ such that $c \cdot H^i_{\mathfrak{m}}(R) = 0$.*

Proof. If $\omega_R^\bullet$ is a normalized dualizing complex for R, it is sufficient (in fact equivalent) to find $0 \neq c$ such that $c \cdot h^{-i}(\omega_R^\bullet) = 0$ by local duality (and the fact that $\mathrm{Hom}_R(\_\_, E)$ is faithful on the modules in question). Now, $h^{-i}(\omega_R^\bullet)$ is finitely generated so it suffices to show that it is not supported everywhere. Now, if we localize at the unique minimal prime Q, we end up with a dualizing complex on a field, which lives in exactly one degree. This degree must be $-\dim R$ by the previous lemma, and so all the other $h^{-i}(\omega_R^\bullet)$ are not supported everywhere as claimed. □

Note that $\mathrm{Ann}_R H^{\dim R}(R) = 0$ and so the above is about as good as one can do. Note a version of the above also holds for non-domains (you can pick c not in any minimal prime defining a maximal component of $\mathrm{Spec}\, R$).

1. F-regularity, a quick way to prove that rings are Cohen-Macaulay

Historically, F-splittings were used to prove lots of rings were Cohen-Macaulay. In modern times, we have learned some really slick ways to prove that integral domains are Cohen-Macaulay.

Definition 1.1. An F-finite ring is called *strongly F-regular* if for every $c \in R$ not contained in any minimal prime, there exists an $e > 0$ such that the map $R \longrightarrow F^e_* R \xrightarrow{F^e_* \cdot c} F^e_* R$ splits as a map of R-modules.

Remark 1.2. Strongly F-regular rings are now known to be the characteristic $p > 0$ analog of rings with KLT singularities in characteristic zero.

Theorem 1.3. *A strongly F-regular Noetherian local domain is Cohen-Macaulay.*

Proof. Fix some $i < \dim R$, we shall show that $H^i_{\mathfrak{m}}(R) = 0$. By Corollary 0.7, we can choose $0 \neq c \in R$ such that $c \cdot H^i_{\mathfrak{m}}(R) = 0$. Now choose an $e > 0$ so that $R \longrightarrow F^e_* R \xrightarrow{F^e_* \cdot c} F^e_* R$ splits as a map of R-modules. Applying $H^i_{\mathfrak{m}}(\_\_)$ we see that

$$H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(F^e_* R) \xrightarrow{F^e_* \cdot c} H^i_{\mathfrak{m}}(F^e_* R)$$

is also injective. But $H^i_{\mathfrak{m}}(F^e_* R) = F^e_* H^i_{\mathfrak{m}^{[p^e]}}(R) = F^e_* H^i_{\mathfrak{m}}(R)$ and so $F^e_* c$ kills it. Thus we have an injective map that is also the zero map, and so the source is zero as claimed. □

Remark 1.4. The *domain* hypothesis is not necessary above, indeed strongly F-regular local rings are normal as we will see shortly, which then implies that they are domains.

This is not so impressive unless you can show that various rings are strongly F-regular. Here are some common ways to prove that rings are strongly F-regular.

Proposition 1.5. *If $(R, \mathfrak{m})$ is a Noetherian local F-finite regular domain, then R is strongly F-regular.*

Proof. Fix $0 \neq c \in R$. Choose $e \gg 0$ so that $c \notin \mathfrak{m}^{[p^e]}$ and so $F^e_* c \notin \mathfrak{m} \cdot F^e_* R = F^e_* \mathfrak{m}^{[p^e]}$. Since a basis for $(F^e_* R)/\mathfrak{m}$ becomes a minimal generating set and hence a basis for the free module $F^e_* R$ (here we use that R is regular), we see that $F^e_* c$ is part of a basis for $F^e_* R$ over R. Thus we can project from $F^e_* c$ to R which produces the map we wanted. □

Proposition 1.6. *Suppose that $R \subseteq S$ is an inclusion of Noetherian domains such that $S \cong R \oplus M$ as R-modules. Then if S is strongly F-regular, so is R.*

Proof. Choose $0 \neq c \in R$. Since S is strongly F-regular, there exists a $\phi : F^e_* S \longrightarrow S$ such that $\phi(F^e_* c) = 1$. Let $\rho : S \longrightarrow R$ be such that $\rho(1_S) = 1_R$ (this exists since $S \cong R \oplus M$). Then the composition $F^e_* R \subset F^e_* S \xrightarrow{\phi} S \xrightarrow{\rho} R$ sends $F^e_* c$ to 1 which proves that R is strongly F-regular. □

Remark 1.7. The above is an open problem in characteristic zero for KLT singularities.

References

[Har66] R. Hartshorne: *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR0222093 (36 #5145)

[Sta16] T. Stacks Project Authors: *stacks project*, 2016.