NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 10TH, 2017

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1. Frobenius splittings of non-normal rings

Suppose that R is not normal but that it is F-split (this isn't impossible, the node is F-split by Fedder's criterion, although the cusp is not).

Lemma 1.1. *Suppose that R is an F-finite reduced Noetherian ring with normalization R^{N}. Further suppose that $\phi : F^e_* R \longrightarrow R$ is an R-linear map. Then ϕ extends to $\phi_{K(R)} : F^e_* K(R) \longrightarrow K(R)$ which restricts to $\phi_{R^{\mathrm{N}}} : F^e_* R^{\mathrm{N}} \longrightarrow R^{\mathrm{N}}$.*

Proof. The extension is easy, simply consider the image $\phi \in \mathrm{Hom}_R(F^e_* R, R) \longrightarrow \mathrm{Hom}_R(F^e_* R, R) \otimes_R K(R) \cong \mathrm{Hom}_R(F^e_* K(R), K(R))$. It is easy to verify that this image $\phi_{K(R)}$ extends ϕ. By restriction we then get a map $\phi_{R^{\mathrm{N}}} : F^e_* R^{\mathrm{N}} \longrightarrow K(R)$. We just need to show that the image lies in R^{N}. Choose $x \in R^{\mathrm{N}}$ and consider $\phi_{R^{\mathrm{N}}}(F^e_* x)$.

Let Q_i denote the minimal primes of R. It is not hard to see that $\phi(F^e_* Q_i) \subseteq Q_i$ for all i (simply localize that Q_i which turns ϕ into a map on the field level, which sends 0 to 0) and so we have induced maps $\phi_i : F^e_*(R/Q_i) \longrightarrow R/Q_i$. But since $R^{\mathrm{N}} = \prod_i (R/Q_i)^{\mathrm{N}}$, it suffices to assume that R is a domain. Let $\mathfrak{c}$ denote the conductor of R in R^{N}. Consider $\mathfrak{c} \cdot \phi_{R^{\mathrm{N}}}(F^e_* R)$. For $z \in \mathfrak{c}$ and any integer $m > 0$

$$\begin{array}{rl} & z \cdot (\phi_{R^{\mathrm{N}}}(F^e_* x))^m \\ = & z \cdot (\phi_{R^{\mathrm{N}}}(F^e_* x)) \cdot (\phi_{R^{\mathrm{N}}}(F^e_* x))^{m-1} \\ = & \phi_{R^{\mathrm{N}}}(F^e_* z^{p^e} x) \cdot (\phi_{R^{\mathrm{N}}}(F^e_* x))^{m-1} \\ \in & \phi_{R^{\mathrm{N}}}(F^e_* \mathfrak{c})(\phi_{R^{\mathrm{N}}}(F^e_* x))^{m-1} \\ \subseteq & \mathfrak{c} \\ \subseteq & R \end{array}$$

In other words if $y = \phi_{R^{\mathrm{N}}}(F^e_* x)$, then $z \cdot y^m \in R$ for all $m > 0$. The result then follows from the following lemma. □

Lemma 1.2. *If R is a Noetherian normal domain with $y \in K(R)$ there exists $0 \neq c \in R$ such that $cy^m \in R$ for all $m > 0$, then $y \in R$.*

Proof. Consider the ideal $I = \langle c, cy, cy^2, cy^3 \ldots \rangle \subseteq R$. Since R is Noetherian, $I = \langle c, cy, \ldots, cy^n \rangle$ for some $n > 0$. Thus we can write $cy^{n+1} = a_n cy^n + \cdots + a_0 c$ for some $a_i \in R$. Dividing by c we obtain that

$$y^{n+1} = a_n y^n + \cdots + a_0$$

which proves that y is integral over R and thus $y \in R$. □

Remark 1.3. The converse to the above lemma holds too, exercise!

Corollary 1.4. *If R is F-split, so is R^{N}, $R/\mathfrak{c}$ and $R^{\mathrm{N}}/\mathfrak{c}$. In particular both $R/\mathfrak{c}$ and $R^{\mathrm{N}}/\mathfrak{c}$ are reduced rings.*

While F-split rings are not normal, they are something called weakly normal.

We'll see this after break, for the rest of the class we'll continue to work on the worksheet on Serre's conditions.

References