NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 20TH, 2017

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1. Frobenius splittings of non-normal rings continued

While F-split and F-injective rings are not necessarily normal, they are something called weakly normal.

Definition 1.1. Suppose R is a reduced Noetherian ring R with finite normalization R^{N}. An extension of rings $R \subseteq R' \subseteq R^{\mathrm{N}}$ is called *subintegral* if $\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R$ is a homeomorphism and if $Q' \in \operatorname{Spec} R'$ then $k(Q' \cap R) \longrightarrow k(Q')$ is an isomorphism. R is called *seminormal* if the only subintegral extension of R is $R' = R$.

An extension of rings $R \subseteq R' \subseteq R^{\mathrm{N}}$ is called *weakly subintegral* if $\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R$ is a homeomorphism and if $Q' \in \operatorname{Spec} R'$ then $k(Q' \cap R) \longrightarrow k(Q')$ is inseparable. R is called *weakly normal* if the only weakly subintegral extension of R is $R' = R$.

We state some facts about weak and semi-normalization without proof.

Lemma 1.2. *Suppose that R is an excellent Noetherian domain.*

- *The seminormalization of R exists. In other words there is a unique subintegral extension $R \subseteq R^{\mathrm{SN}} \subseteq R^{\mathrm{N}}$ with R^{SN} seminormal*
- *The formation of the seminormalization commutes with localization. In particular if R is seminormal so are its localizations.*
- *The weak normalization of R exists. In other words there is a unique weakly subintegral extension $R \subseteq R^{\mathrm{WN}} \subseteq R^{\mathrm{N}}$ with R^{WN} weakly normal.*
- *The formation of the weak normalization commutes with localization. In particular if R is weakly normal so are its localizations.*

Our goal for now is to show that F-injective rings (and hence F-split rings) are weakly normal. First we give another characterization of weakly normal rings.

Proposition 1.3. *Suppose that R is a reduced Noetherian ring of characteristic $p > 0$ with $R \subseteq R^{\mathrm{N}}$ finite. Then the following are equivalent.*

(a) *$x \in K(R)$ and $x^p \in R$ implies that $x \in R$.*
(b) *R is weakly normal.*

Proof. We first show that (a) $\Rightarrow$ (b). Suppose that R is not weakly normal, this means that there exists $R \subsetneq R'$ weakly subintegral. By localizing, we may assume that $(R, \mathfrak{m}, k)$ is weakly normal except at $\mathfrak{m}$ and so $(R', \mathfrak{m}', k')$ is local as well. Choose some $x \in R'$ which we will try to show is in R. Let $\mathfrak{c}$ be the conductor of R'/R and note it is $\mathfrak{m}$-primary by assumption (and also $\mathfrak{m}'$-primary in R'). It is easy to see that R is the gluing of $(R' \longrightarrow R'/\mathfrak{c} \leftarrow R/\mathfrak{c})$. Now, there are two possibilities.

(1) $x \in \mathfrak{m}'$. In this case $x^{p^e} \in \mathfrak{c}$ for some e. But $\mathfrak{c} \subseteq R$ and this case is taken careof.

(2) x is a unit in R' and so consider $\overline{x} \in R'/\mathfrak{m}' = k'$. Thus $\overline{x^{p^e}} \in k$ for some $e > 0$ since $k \subseteq k'$ is purely inseparable. Consider $y \in R$ with the same image in k. It follows that $z = x^{p^e} - y \in \mathfrak{m}'$ and so applying (1) to z, we see that $z \in R$. But then $x^{p^e} = z + y \in R$ as well. But now $x \in R$ again.

In either case, $x \in R$.

Now we prove that (b) $\Rightarrow$ (a). Choose $x \in K(R)$ with $x^p \in R$. Consider the extension $R \subseteq R[x]$. It suffices to prove that this is weakly subintegral. Since we have $R \subseteq R[x] \subseteq R^{1/p}$ are all integral extensions, we see that $\operatorname{Spec} R[x] \longrightarrow \operatorname{Spec} R$ is a bijection. On the other hand for each $Q' \in \operatorname{Spec} R[x]$ with $Q = R \cap Q'$, we see that $k(Q) \subseteq k(Q') \subseteq k(Q)^{1/p}$ by the above factorization. Thus $k(Q) \subseteq k(Q')$ is purely inseparable and so $R \subseteq R[x]$ is weakly subintegral as claimed. □

We need one more lemma before proving our result on weak normality of F-injective rings.

Lemma 1.4. *Suppose that (R, m) is a reduced local ring of characteristic p, $X = \operatorname{Spec} R$ and that $X - m$ is weakly normal. Then X is weakly normal if and only if the action of Frobenius is injective on $H^1_m(R)$.*

Proof. We assume that the dimension of R is greater than 0 since the zero-dimensional case is trivial. Embed R in its weak normalization $R \subset R^{\mathrm{WN}}$ (which is of course an isomorphism outside of m). We have the following diagram of R-modules.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R & \hookrightarrow & \Gamma(X - m, \mathscr{O}_{X-m}) & \twoheadrightarrow & H^1_m(R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & R^{\mathrm{WN}} & \hookrightarrow & \Gamma(X^{\mathrm{wn}} - m, \mathscr{O}_{X^{\mathrm{wn}}-m}) & \twoheadrightarrow & H^1_m(R^{\mathrm{WN}}) & \longrightarrow & 0
\end{array}
$$

The left horizontal maps are injective because R and ${}^*R$ are reduced. One can check that Frobenius is compatible with all of these maps. Now, R is weakly normal if and only if R is weakly normal in R^{WN} if and only if every $r \in R^{\mathrm{WN}}$ with $r^p \in R$ also satisfies $r \in R$ by Proposition 1.3.

First assume that the action of Frobenius is injective on $H^1_m(R)$. So suppose that there is such an $r \in R^{\mathrm{WN}}$ with $r^p \in R$. Then r has an image in $\Gamma(X - m, \mathscr{O}_{X-m})$ and therefore an image in $H^1_m(R)$. But r^p has a zero image in $H^1_m(R)$, which means r has zero image in $H^1_m(R)$ which guarantees that $r \in R$ as desired.

Conversely, suppose that R is weakly normal. Let $r \in \Gamma(X - m, \mathscr{O}_{X-m})$ be an element such that the action of Frobenius annihilates its image $\overline{r}$ in $H^1_m(R)$. Since $r \in \Gamma(X - m, \mathscr{O}_{X-m})$ we identify r with a unique element of the total field of fractions of R. On the other hand, $r^p \in R$ so $r \in {}^*R = R$. Thus we obtain that $r \in R$ and so $\overline{r}$ is zero as desired. □

Theorem 1.5. *Suppose that R is a reduced F-finite F-injective Noetherian ring. Then R is weakly normal.*

Proof. It is not difficult to verify that weak normality can be checked locally and so suppose that $(R, \mathfrak{m})$ is a local ring. Also recall that if Q is any prime of R then R_Q is also F-injective by the worksheet (here we use that R is F-finite). Now we need to show that R is weakly normal. If R is not weakly normal, choose a prime $P \in \operatorname{Spec} R$ of minimal height with respect to the condition that R_P is not weakly normal. Apply 1.4 to get a contradiction. □

References