NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 22ND, 2017

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1. A quick introduction to $\mathbb{Q}$-Weil divisors

Setting 1.1. Throughout this section R is a normal Noetherian domain.

We first state some facts about S2 and reflexive modules.

Definition 1.2. A finitely generated R-module M is called *reflexive* if the canonical map $M \mapsto \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is an isomorphism. Given M, the *reflexification of* M is simply $\mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R) =: M^{\vee\vee}$.

Exercise 1.1. Show that for any finitely generated module M, $\mathrm{Hom}_R(M, R)$ is reflexive.

Lemma 1.3. *A finitely generated R-module M is reflexive if and only if it is* $\mathbf{S}_2$.

Proof. We leave at as an exercise, see [Har94] for details. □

Example 1.4. If M is a torsion-free R-module of rank-1 (meaning that $M \otimes_R K(R) \cong K(R)$), then because M is torsion free, the canonical map $M \longrightarrow M \otimes_R K(R)$ is injective. Thus we can embed $M \subseteq K(R)$. In this case, we see that $\mathrm{Hom}_R(M, R) \cong R :_{K(R)} M$. Indeed, any such $a \in R :_{K(R)} M$ yields a homomorphism by multiplication. Conversely, given any $\phi \in \mathrm{Hom}_R(M, R) \subseteq \mathrm{Hom}_{K(R)}(M \otimes_R K(R), K(R))$ and then our identification $M \otimes_R K(R) \cong K(R)$ lets us identify ϕ with multiplication by some element of $K(R)$.

In this case, $M^{\vee\vee}$, the reflexification of M, can be viewed as $R :_{K(R)} (R :_{K(R)} M)$. This is a subset of $K(R)$ that obviously contains M.

Definition 1.5. A Weil divisor $D = \sum a_i D_i$ on $\mathrm{Spec}\, R$ (or on R) is a finite formal $\mathbb{Z}$-sum of distinct height one prime ideals D_i. A $\mathbb{Q}$-(Weil-)divisor $D = \sum a_i D_i$ on $\mathrm{Spec}\, R$ is a finite formal $\mathbb{Q}$-sum of distinct height one prime ideals D_i. In either case, the divisor is called *effective* if all the $a_i \geq 0$.

Given any $0 \neq g \in K(R)$, we define $\mathrm{div}(g) = \sum v_{D_i}(g) D_i$ where $v_{D_i}(g)$ is the value of g with respect to the discrete valuation v_{D_i} which one obtains after localizing R at D_i.

Associated to any Weil divisor D is a reflexive fractional ideal[1] $R(D)$ (frequently denoted in the sheaf theory language as $\mathscr{O}_{\mathrm{Spec}\, R}(D)$). In particular, if $D = \sum a_i D_i$ then $R(D)$ is the subset of $K(R)$ that have poles of order at most a_i at D_i whenever $a_i > 0$ and have zeros of order at least $|a_i|$ at D_i whenever $a_i < 0$. Explicitly,

$$R(D) = \{g \in K(R) \mid \mathrm{div}(g) + D \geq 0\}.$$

Exercise 1.2. Using the definition, show that $R(D)$ is reflexive, or equivalently that it is $\mathbf{S}_2$.

Let's say what this is explicitly in some special cases.

[1] A fractional ideal is by definition a finitely generated submodule of $K(R)$.

(i) If $D = 0$, then $R(D) = R$.
(ii) If $D = D_i$ is a single prime ideal, then $R(D) := R :_{K(R)} D$.
(iii) If $D = -D_i$ is the negative of a single prime, then $R(D) = D_i$.
(iv) If $D = -nD_i$ (for $n \geq 0$) is the negative of a single divisor, then $R(D) = (D_i^n)^{\vee\vee}$.
(v) If $D = -\sum a_i D_i$ (for $a_i \geq 0$), then $R(D) = \left(\prod D_i^{a_i}\right)^{\vee\vee}$.
(vi) If $D \geq 0$ (is effective) then $R(D) = R :_{K(R)} R(-D)$.
(vii) If $D = A + B$, then $R(D) = \big(R(A) \cdot R(B)\big)^{\vee\vee}$.
(viii) For any D, $R(-D) = R :_{K(R)} R(D)$.
(ix) If $D = A - B$ then $R(D) \cong \operatorname{Hom}_R(R(B), R(A)) \cong R(A) :_{K(R)} R(B)$.
(x) For any $0 \neq f, g \in K(R)$, $-\operatorname{div}(g) = \operatorname{div}(1/g)$ and also $\operatorname{div}(f \cdot g) = \operatorname{div}(f) + \operatorname{div}(g)$.
(xi) $R(\operatorname{div}(g)) = \frac{1}{g} \cdot R$.

Definition 1.6 (Cartier divisors)**.** A Weil divisor D is called *Cartier* if $R(D)$ is projective (locally free). A ($\mathbb{Q}$-)divisor D is called $\mathbb{Q}$-*Cartier* if there exists an integer $n > 0$ such that nD is Cartier.

Example 1.7. In $k[x^2, xy, y^2]$ the ideal $Q = \langle x^2, xy \rangle$ corresponds to a prime divisor D but D is not Cartier (since it is not generated by a single element locally). However, D is $\mathbb{Q}$-Cartier since $R(-2D) = \langle x^2 \rangle$ (and hence $R(2D) = \frac{1}{x^2}R$).

Definition 1.8 (Linear equivalence)**.** Two Weil divisors D_1, D_2 are said to be *linearly equivalent* if $D_1 - D_2 = \operatorname{div}(g)$ for some $0 \neq g \in K(R)$. In this case we write $D_1 \sim D_2$. If D_1, D_2 are $\mathbb{Q}$-divisors, we say that they are $\mathbb{Q}$-*linearly equivalent* if there exists an integer $n > 0$ such that nD_1 and nD_2 are linearly equivalent Weil divisors.

Example 1.9. Working in $k[x^2, xy, y^2]$ set D_1 to be the prime divisor $\langle x^2, xy \rangle$ and D_2 to be the prime divisor $\langle xy, y^2 \rangle$, then $D_1 - D_2 = \operatorname{div}(x/y)$ and so $D_1 \sim D_2$.

Lemma 1.10. *Two divisors D_1 and D_2 are linearly equivalent if and only if there is an (abstract) isomorphism $R(D_1) \cong R(D_2)$.*

Proof. Suppose first that D_1 and D_2 are linearly equivalent, and so $D_1 - \operatorname{div}(g) = D_2$ for some $0 \neq g \in K(R)$. We claim that

$$R(D_1) \cdot g = R(D_2).$$

Choose $f \in R(D_1)$. Then $\operatorname{div}(f) + D_1 \geq 0$. It follows that $\operatorname{div}(f \cdot g) + D_1 = \operatorname{div}(f) + D_1 + \operatorname{div}(g) \geq \operatorname{div}(g)$. Thus $\operatorname{div}(f \cdot g) + D_1 - \operatorname{div}(g) = \operatorname{div}(f \cdot g) + D_2 \geq 0$ and so $f \cdot g \in R(D_2)$. Conversely, if $h \in R(D_2)$ then $\operatorname{div}(h) + D_2 \geq 0$ and so $0 \leq \operatorname{div}(h) + D_1 - \operatorname{div}(g) = \operatorname{div}(h/g) + D_1$ which implies that $h/g \in R(D_1)$ and so $h \in R(D_1) \cdot g$ as desired.

Conversely, suppose that $R(D_1) \cong R(D_2)$ and so $\operatorname{Hom}_R(R(D_1), R(D_2)) \cong R$. Since we have $R(D_1), R(D_2) \subseteq K(R)$ we see that $R(D_2) :_{K(R)} R(D_1) = h \cdot R$ for some $h \in K(R)$. We see that $h \cdot R(D_1) = R(D_2)$ and so an argument similar to the one above shows that $D_1 - D_2 = \operatorname{div}(h)$. □

Lemma 1.11. *If D is a divisor on R, then every $g \in R(D)$ determines an effective divisor $D_g \sim D$, explicitly $D_g := D + \operatorname{div}(g)$. Furthermore $h \in R(D)$ determines the same divisor as g if and only if h and g are associates in R (unit multiplies).*

References

[Har94] R. Hartshorne: *Generalized divisors on Gorenstein schemes*, Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992), vol. 8, 1994, pp. 287–339. MR1291023 (95k:14008)