NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 24TH, 2017

KARL SCHWEDE

1. A quick introduction to $\mathbb{Q}$-Weil divisors continued

Lemma 1.1. *If D is a divisor on R, then every $g \in R(D)$ determines an effective divisor $D_g \sim D$, explicitly $D_g := D + \mathrm{div}(g)$. Furthermore $h \in R(D)$ determines the same divisor as g if and only if h and g are associates in R (unit multiplies).*

Proof. We first observe that $\mathrm{div}(g) = \mathrm{div}(h)$ if and only if $\mathrm{div}(g/h) = 0$. But $\mathrm{div}(g/h) = 0$ if and only if g/h has zero valuation at each height one prime of R. Obviously units have this property. On the other hand if $\mathrm{div}(g/h) = 0$, then $g/h \in R$ and must also be a unit (because if not, it would vanish at some height one prime). This handles the uniqueness. Now we simply have to show that $D_g \geq 0$. But $R(D)$ is the set of elements $g \in K(R)$ such that $\mathrm{div}(g) + D \geq 0$. □

Remark 1.2. The choice of a finitely generated reflexive rank-1 module M and an embedding $M \subseteq K(R)$ also determines a divisor D such that $M = R(D)$. To see this, for each height one prime D_i with associated discrete valuation v_{D_i} set $a_i = \max\{-v_{D_i}(m) \mid 0 \neq m \in M\}$. It can be verified that $D = \sum a_i D_i$ is a divisor (one has to verify that the sum is finite using that M is finitely generated) and that $M \subseteq R(D)$. By construction $R(D) \to M$ is an isomorphism at height one primes and so since both modules are $\mathbf{S}_2$, it is an isomorphism everywhere.

With this in mind, I like to think about a choice of $g \in R(D)$ as a choice of a new way to embed $R(D)$ into $K(R)$. In particular, it is just the embedding of $R(D)$ into $K(R)$ which sends g to 1.

Suppose now that R has a canonical module ω_R.

Lemma 1.3. *ω_R is $\mathbf{S}_2$.*

Proof. Note the canonical module is unique up to tensoring with a locally free module, which obviously does not change whether a module is $\mathbf{S}_2$, so any canonical module is as good as any other. The Lemma can be checked after localization and completion (since it is a statement about local cohomology and the completion of a dualizing complex is a dualizing complex say by local duality). Thus choose $A \subseteq R$ a finite extension with A complete and regular (this exists as part of the Cohen Structure Theorem, see for example [Sta16, Tag 032D]). It follows that $\mathbf{R}\mathrm{Hom}_A(R, A)$ is a dualizing complex for R. A canonical module for R is thus $h^0 \mathbf{R}\mathrm{Hom}_A(R, A) = \mathrm{Hom}_A(R, A)$. Now, $\mathrm{Hom}_A(R, A)$ is obviously reflexive as an A-module and hence it is $\mathbf{S}_2$ as an A-module. But then it is not hard to see that $\mathrm{Hom}_A(R, A)$ is $\mathbf{S}_2$ as an R-module as well (this requires a bit of work). Hence $\omega_R = \mathrm{Hom}_A(R, A)$ is reflexive as claimed. □

Definition 1.4. Fix ω_R a canonical module. A *canonical divisor* is any Weil divisor K_R such that $\omega_R \cong R(K_R)$.

Remark 1.5. So far we seen that canonical modules are only unique up to twisting by locally free modules. In particular, based on what we have seen, canonical modules are only unique up to a Cartier divisor. For local rings this is file (all Cartier divisors are linearly equivalent to zero). For more general rings and especially for schemes, this is not so good since you want some compatibility between your dualizing complexes (you want them to behave reasonably with respect to morphisms). However, most of the time we are working with objects and morphisms (essentially) of finite type over a Gorenstein ring A (for example a field or $\mathbb{Z}_p$), say $f : X \longrightarrow \operatorname{Spec} A$. In that case, there is a canonical choice of a dualizing complex, $f^! A[\dim A]$. In the case that R is a normal domain of finite type over a field k, $R = k[x_1, \dots, x_n]/I$, then this choice boils down to $\omega_R = \operatorname{Ext}^{n-\dim R}_{k[x_1,\dots,x_n]}(R, k[x_1, \dots, x_n])$.

Note that a ring is Gorenstein if and only if K_R is Cartier and R is Cohen-Macaulay. We also have the following definition.

Definition 1.6 ($\mathbb{Q}$-Gorenstein)**.** R is called $\mathbb{Q}$-*Gorenstein* if K_R is $\mathbb{Q}$-Cartier. (Note there is no Cohen-Macaulay hypothesis).

2. Frobenius splittings and divisors

Suppose that R is an F-finite Noetherian normal domain. Notice that $\operatorname{Hom}_R(F^e_* R, R)$ is a reflexive R-module and hence a $\mathbf{S}_2$ R-module. Since the $\mathbf{S}_2$ condition can be checked via local cohomology and localization, neither of which care whether we are viewing the Hom-set as an R-module or $F^e_* R$-module, it follows that $\operatorname{Hom}_R(F^e_* R, R)$ is $\mathbf{S}_2$ as an $F^e_* R$-module as well.

Exercise 2.1. Suppose R is an F-finite Noetherian normal domain, show carefully that the $F^e_* R$-module $\operatorname{Hom}_R(F^e_* R, R)$ is $\mathbf{S}_2$.

Thus $\operatorname{Hom}_R(F^e_* R, R)$ is a reflexive $F^e_* R$-module of rank 1 (it has rank 1 because its rank as an R-module is the same as the rank of $F^e_* R$ as an R-module). You might naturally ask what linear equivalence class of divisor this Hom set corresponds to?

First we work in the following setting.

Setting 2.1. Suppose R is as above in this section. Suppose we have a dualizing complex $\omega^{\bullet}_R$ such that $\mathbf{R}\operatorname{Hom}_R(F^e_* R, \omega^{\bullet}_R) \cong \omega^{\bullet}_{F^e_* R}$ (this always exists if R is local or of finite type over a field, it probably also follows that such a dualizing complex exists in general by some unpublished work). We fix this dualizing complex forever more. Notice that if ω_R is the associated canonical module, the $\operatorname{Hom}_R(F^e_* R, \omega_R) \cong \omega_{F^e_* R} \cong F^e_* \omega_R$ (note we don't need to worry about the derived Hom's all the modules are reflexive and they are certainly isomorphic in codimension 1 where R is regular). We fix K_R to be a canonical divisor associated to this canonical module.

Lemma 2.2. *With notation above, $F^e_* R((1 - p^e)K_R) \cong \operatorname{Hom}_R(F^e_* R, R)$. In particular, if R is local and quasi-Gorenstein, then $\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* R$ as $F^e_* R$-modules.*

Proof. This follows from the following chain of isomorphisms (in this chain, in almost every step, we use that all modules are reflexive, and so it is enough to verify the isomorphisms in codimension 1 where we can treat the modules as if they were free).

$$\begin{array}{rl} & \operatorname{Hom}_R(F^e_* R, R) \\ \cong & \operatorname{Hom}_R((F^e_* R) \otimes_R R(K_R), R(K_R)) \\ \cong & \operatorname{Hom}_R(F^e_*(R \otimes_R R(p^e K_R)), \omega_R) \\ \cong & \operatorname{Hom}_{F^e_* R}(F^e_*(R \otimes_R R(p^e K_R)), \operatorname{Hom}_R(F^e_* R, \omega_R)) \\ \cong & \operatorname{Hom}_{F^e_* R}(F^e_*(R(p^e K_R)), F^e_* \omega_R) \\ \cong & F^e_* \operatorname{Hom}_R((R(p^e K_R)), R(K_R)) \\ \cong & F^e_* R((1-p^e)K_R). \end{array}$$

□

Corollary 2.3. *Every nonzero map $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ determines an effective Weil divisor $D_\phi \sim (1-p^e)K_R$. Furthermore, two maps ϕ, ϕ' determine the same divisor if and only if they are the same up to pre-multiplication by a unit of $F^e_* R$.*

References

[Sta16] T. Stacks Project Authors: *stacks project*, 2016.