NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 27TH, 2017

KARL SCHWEDE

1. Frobenius splittings and divisors continued

Recall we work in the following setting.

Setting 1.1. Suppose R is as above in this section. Suppose we have a dualizing complex $\omega_R^{\bullet}$ such that $\mathbf{R}\operatorname{Hom}_R(F^e_* R, \omega_R^{\bullet}) \cong \omega^{\bullet}_{F^e_* R}$ (this always exists if R is local or of finite type over a field, it probably also follows that such a dualizing complex exists in general by some unpublished work). We fix this dualizing complex forever more. Notice that if ω_R is the associated canonical module, the $\operatorname{Hom}_R(F^e_* R, \omega_R) \cong \omega_{F^e_* R} \cong F^e_* \omega_R$ (note we don't need to worry about the derived Hom's all the modules are reflexive and they are certainly isomorphic in codimension 1 where R is regular). We fix K_R to be a canonical divisor associated to this canonical module.

Also recall the following from last time.

Lemma 1.2. *With notation above, $F^e_* R((1-p^e)K_R) \cong \operatorname{Hom}_R(F^e_* R, R)$. In particular, if R is local and quasi-Gorenstein, then $\operatorname{Hom}_R(F^e_* R, R) \cong F^e_* R$ as $F^e_* R$-modules.*

Corollary 1.3. *Every nonzero map $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ determines an effective Weil divisor $D_\phi \sim (1-p^e)K_R$. Furthermore, two maps ϕ, ϕ' determine the same divisor if and only if they are the same up to pre-multiplication by a unit of $F^e_* R$.*

Corollary 1.4. *Suppose R is a normal Noetherian F-finite domain. There exists a $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ which generates the* Hom*-set as an $F^e_* R$-module if and only if $(1-p^e)K_R \sim 0$. In the case that R is local, such a ϕ exists for some $e > 0$ if and only if R is $\mathbb{Q}$-Gorenstein with index not divisible by $p > 0$.*

In many cases you want this divisor to be in some sense independent of the characteristic, or more generally, independent of self-composition. In particular, you'd like the divisor corresponding to $\phi \circ F^e_* \phi$ to be the same as the divisor corresponding to ϕ. We can accomplish this by normalizing our divisor.

Definition 1.5. For any nonzero $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ we define $\Delta_\phi := \frac{1}{p^e-1} D_\phi$. Note that $K_R + \Delta_\phi \sim_{\mathbb{Q}} 0$.

Lemma 1.6. *If $\Phi \in \operatorname{Hom}_R(F^e_* R, R)$ generates the module (as an $F^e_* R$-module), then $\Delta_\Phi = 0 = D_\Phi$. Furthermore, if we write $\phi(F^e_* \_\_) = \psi(F^e_*(r \cdot \_\_))$ for some $r \in R$, then $D_\phi = D_\psi + \operatorname{div}_R(r)$ and so $\Delta_\phi = \Delta_\psi + \frac{1}{p^e-1}\operatorname{div}_R(r)$.*

Proof. Left as an exercise to the reader. □

Lemma 1.7. *For any map ϕ and any integer n form $\phi^n := \phi \circ (F^e_* \phi) \circ (F^{2e}_* \phi) \circ \cdots \circ (F^{(n-1)e}_* \phi) \in \operatorname{Hom}_R(F^{ne}_* R, R)$. Then $\Delta_{\phi^n} = \Delta_\phi$.*

Proof. This statement may be verified in codimension 1 since divisors are defined in codimension 1. Thus we localize R at a height one prime to obtain the $(R, \mathfrak{m} = \langle r \rangle)$ is a DVR (remember, R was normal). Since regular rings are Gorenstein, we choose $\Phi \in \mathrm{Hom}_R(F^e_* R, R)$ generating the Hom set as an $F^e_* R$-module. Then we can write $\phi(F^e_* \_\_) = \Phi(F^e_* u r^n \_\_)$ for some integer $n > 0$ and unit $u \in R$. Note that in this case, $D_\phi = n \operatorname{div}(r)$ and so $\Delta_\phi = \frac{n}{p^e - 1} \operatorname{div}(r)$. It follows that

$$\phi^2(F^{2e}_* \_\_) = \Phi^2(F^{2e}_* (u r^n)^{1+p^e} \_\_)$$

and so $\Delta_{\phi^2} = \frac{n(1+p^e)}{p^{2e}-1} \operatorname{div}(r) = \frac{n}{p^e - 1} \operatorname{div}(r) = \Delta_\phi$. More generally

$$\phi^n(F^{ne}_* \_\_) = \Phi^n(F^{ne}_* (u r^n)^{1+p^e+\dots+p^{(n-1)e}} \_\_)$$

and thus $\Delta_{\phi^n} = \frac{n(1+p^e+\dots+p^{(n-1)e})}{p^{ne}-1} \operatorname{div}(r) = \frac{n}{p^e-1} \operatorname{div}(r) = \Delta_\phi$ □

Exercise 1.1. Suppose that $0 \neq \phi \in \mathrm{Hom}_R(F^e_* R, R)$ and $0 \neq \psi \in \mathrm{Hom}_R(F^d_* R, R)$. Find a formula for $\Delta_{\phi \circ F^e_* \psi}$ in terms of Δ_ϕ and Δ_ψ.

Putting together what we know now, we have a bijection

$$\left\{ \begin{array}{c} \mathbb{Q}\text{-divisors } \Delta \text{ such that} \\ K_R + \Delta \sim_{\mathbb{Q}} 0 \\ \text{with trivializing index}^1 \text{ not} \\ \text{divisible by } p \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Nonzero} \\ \phi \in \mathrm{Hom}_R(F^e_* R, R) \end{array} \right\} \Big/ \sim$$

where the equivalence on the right is generated by self-composition and pre-multiplication by units.

Suppose that $\Delta = \Delta_\phi$ is such that $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ satisfies $\phi(F^e_* I) \subseteq I$. For simplicity assume that R/I is normal, then ϕ induces a map $\phi_{R/I} : F^e_*(R/I) \to R/I$ which, if this is not the zero map, induces a divisor $\Delta_{R/I}$. In particular for every such Δ_ϕ with $K_R + \Delta_\phi \sim_{\mathbb{Q}} 0$ we obtain a canonical $\Delta_{R/I}$ with $K_{R/I} + \Delta_{R/I} \sim_{\mathbb{Q}} 0$. This canonical way of associating a divisor on a subscheme is an analog of Kawamata's subadjunction theorem in the characteristic zero world [Kaw98].

References

[Kaw98] Y. Kawamata: *Subadjunction of log canonical divisors. II*, Amer. J. Math. **120** (1998), no. 5, 893–899. MR1646046 (2000d:14020)