NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 29TH, 2017

KARL SCHWEDE

1. Frobenius splittings and divisors, continued

Last time, we learned we had a bijection.

$$\left\{ \begin{array}{c} \mathbb{Q}\text{-divisors } \Delta \text{ such that} \\ K_R + \Delta \sim_{\mathbb{Q}} 0 \\ \text{with trivializing index}^1 \text{ not} \\ \text{divisible by } p \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Nonzero} \\ \phi \in \operatorname{Hom}_R(F^e_* R, R) \end{array} \right\} \Big/ \sim$$

where the equivalence on the right is generated by self-composition and pre-multiplication by units.

Suppose that $\Delta = \Delta_\phi$ is such that $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ satisfies $\phi(F^e_* I) \subseteq I$. For simplicity assume that R/I is normal, then ϕ induces a map $\phi_{R/I} : F^e_*(R/I) \to R/I$ which, if this is not the zero map, induces a divisor $\Delta_{R/I}$. In particular for every such Δ_ϕ with $K_R + \Delta_\phi \sim_{\mathbb{Q}} 0$ we obtain a canonical $\Delta_{R/I}$ with $K_{R/I} + \Delta_{R/I} \sim_{\mathbb{Q}} 0$. This canonical way of associating a divisor on a subscheme is an analog of Kawamata's subadjunction theorem in the characteristic zero world [Kaw98].

Let's do an example of this.

Example 1.1. Suppose $\operatorname{char} k > 2$, consider $S = k[x, y, z]$, $R = k[x, y, z]/\langle xy - z^2\rangle$ and $I = \langle x, z\rangle \subseteq R$. $\operatorname{Hom}_R(F^e_* R, R)$ is isomorphic to $F^e_* R$ and is generated by $\Phi_R(F^e_* \_\_)$, the restriction of the map $\Phi_S(F^e_*(xy - z^2)^{p^e-1} \cdot \_\_) \in \operatorname{Hom}_S(F^e_* S, S)$ (by Fedder's criterion). Now, consider the map

$$\psi(F^e_* \_\_) = \Phi_R(F^e_* x^{(p^e-1)/2} \cdot \_\_)$$

which is the restriction of $\Phi_S(F^e_*(xy - z^2)^{p^e-1} x^{(p^e-1)/2} \cdot \_\_)$. If you apply this map to the ideal $I = \langle x, z\rangle$ one obtains

$$\Phi_S(F^e_*(xy - z^2)^{p^e-1} x^{(p^e-1)/2} \langle x, z\rangle)$$

We claim that $(xy - z^2)^{p^e-1} x^{(p^e-1)/2} \langle x, z\rangle \subseteq \langle x^{p^e}, z^{p^e}\rangle$. Indeed, when we expand we get

$$\big((xy)^{p^e-1} x^{(p^e-1)/2} + \cdots + (xy)^{(p^e-1)/2} z^{(p^e-1)} x^{(p^e-1)/2} + \cdots + z^{2(p^e-1)} x^{(p^e-1)/2}\big)$$

which when multiplied by $\langle x, z\rangle$ is obviously contained in $\langle x^{p^e}, z^{p^e}\rangle$ which means that $\phi_R(F^e_* I) \subseteq I$. On the other hand, if we didn't multiply by $\langle x, z\rangle$ then the expansion is not in $\langle x^{p^e}, z^{p^e}\rangle$ which means that $\psi = \phi_{R/I}$ is not the zero map. In particular it induces a divisor on $k[y] = S/\langle x, z\rangle$ and so we can ask what divisor it gives us.

In the above expansion, the only term that is not in $\langle x^{p^e}, z^{p^e}\rangle$ is the middle term $(xy)^{(p^e-1)/2} z^{(p^e-1)} x^{(p^e-1)/2}$. The generating map for $\operatorname{Hom}(F^e_* S/\langle x, z\rangle, S/\langle x, z\rangle)$ is obtained from $\Phi_S(F^e_* x^{p^e-1} z^{p^e-1} \_\_)$. Hence the map ψ we found is the generating map for $\operatorname{Hom}(F^e_* k[y], k[y])$ pre-multiplied by $y^{(p^e-1)/2}$. In particular, the divisor corresponding to ψ is $\frac{1}{2}\operatorname{div}(y)$ on $\operatorname{Spec} k[y]$.

2. Divisors, Frobenius splittings and finite extensions

Suppose that $R \subseteq S$ is a finite extension of normal Noetherian domains. Suppose we have $\phi : F^e_* R \longrightarrow R$. It is natural to ask when ϕ extends to $\phi_S : F^e_* S \longrightarrow S$ and when it does, what is the relation between the divisors Δ_R and Δ_S.

Example 2.1. Suppose $R = k[x^2] \subseteq k[x] = S$ for $\mathrm{char} k = p > 2$. Consider the generating homomorphism $\Phi \in \mathrm{Hom}_R(F_* R, R)$ which sends $F_*(x^2)^{p-1}$ to 1 and the other basis elements to zero. This map does not extend to a map $\mathrm{Hom}_S(F_* S, S)$, indeed if it did then $F_* x^{p-2} = F_* \frac{x^{2(p-1)}}{x^p}$ would necessarily be sent to $1/x$ which is not in S.

On the other hand, consider the map $\phi \in \mathrm{Hom}_R(F_* R, R)$ is the map that projects onto the basis monomial $F_*(x^2)^{(p-1)/2}$ and projects the other basis monomials to zero. It is easy to see that any extension of this map to S sends $F_* x^{p-1} \mapsto 1$ since $x^{p-1} = (x^2)^{(p-1)/2}$. Furthermore we claim such an extension also sends the other basis monomials to zero. Consider the element $F_* x^j$ for $0 \leq j < p-1$. If j is even, then there is no problem, $F_* x^j$ is one of the other basis elements of $F_* R$ and is thus sent to zero. So suppose j is odd. Now, $j + p$ is even and $j + p \leq (p-1) + p$ but since the right side is odd, $j + p \leq 2(p-1)$. Thus $F_* x^{j+p}$ is a monomial basis element of $F_* R$. But since $j+p$ cannot equal $p - 1 = (2(p-1))/2$, we see that any extension of ϕ must send $F_* x^{j+p}$ to zero. But then such an extension must send $F_* x^j$ to $0/p = 0$. Therefore, because we can describe an extension of ϕ by describing what it does to the basis elements of $F_* S$, we see that ϕ can indeed extend. It follows immediately that any $\psi(F_* \_\_) := \phi(F_*(r \cdot \_\_))$ also extends.

References

[Kaw98] Y. Kawamata: *Subadjunction of log canonical divisors. II*, Amer. J. Math. **120** (1998), no. 5, 893–899. MR1646046 (2000d:14020)