NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 3RD, 2017

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Proposition 0.1. *Suppose that $R \subseteq S$ is an inclusion of Noetherian domains such that $S \cong R \oplus M$ as R-modules. Then if S is strongly F-regular, so is R.*

Proof. Choose $0 \neq c \in R$. Since S is strongly F-regular, there exists a $\phi : F^e_* S \longrightarrow S$ such that $\phi(F^e_* c) = 1$. Let $\rho : S \longrightarrow R$ be such that $\rho(1_S) = 1_R$ (this exists since $S \cong R \oplus M$). Then the composition $F^e_* R \subset F^e_* S \xrightarrow{\phi} S \xrightarrow{\rho} R$ sends $F^e_* c$ to 1 which proves that R is strongly F-regular. □

Remark 0.2. The above is an open problem in characteristic zero for KLT singularities.

Corollary 0.3. *A direct summand of a regular ring in characteristic $p > 0$ is Cohen-Macaulay.*

The above is obvious if we are taking a finite local inclusion of local rings of the same dimension. It is not so obvious otherwise (indeed, it has perhaps only recently been discovered how to show that direct summands of regular rings in mixed characteristic are Cohen-Macaulay).

Proposition 0.4. *If R is an F-finite ring such that $R_{\mathfrak{m}}$ is strongly F-regular for each maximal $\mathfrak{m} \in \operatorname{Spec} R$, then R is strongly F-regular.*

Proof. Obviously strongly F-regular rings are F-split (take $c = 1$) and so R is F-split. By post composing with Frobenius splittings, if we have a map $\phi : F^e_* R \longrightarrow R$ which sends $F^e_* c \mapsto 1$, then we can replace e by a larger e. Now, pick $0 \neq c \in R$. For each $\mathfrak{m} \in \operatorname{Spec} R$, $\operatorname{Hom}_R(F^e_* R, R)_{\mathfrak{m}} \xrightarrow{eval@c} R_{\mathfrak{m}}$ is surjective for $e \gg 0$. But thus for each $\mathfrak{m}$, there exists a neighborhood $U_{\mathfrak{m}}$ of $\mathfrak{m}$ and some $e_{\mathfrak{m}}$ such that $\operatorname{Hom}_R(F^e_* R, R)_{\mathfrak{n}} \xrightarrow{eval@c} R_{\mathfrak{n}}$ surjects for all $\mathfrak{n} \in U_{\mathfrak{m}}$ where we take $e = e_{\mathfrak{m}}$. These $U_{\mathfrak{m}}$ cover $\operatorname{Spec} R$ and since $\operatorname{Spec} R$ is quasi-compact, we may pick finitely many of them, choose a common large enough $e > 0$ and observe that $\operatorname{Hom}_R(F^e_* R, R) \xrightarrow{eval@c} R$ surjects as desired. □

Theorem 0.5. *Suppose that $0 \neq d \in R$ is such that $R[d^{-1}]$ is strongly F-regular and such that there exists a map $\phi : F^e_* R \longrightarrow R$ satisfying $\phi(F^e_* d) = 1$. Then R is strongly F-regular.*

Proof. Note first that R is F-split (pre-multiply ϕ by $F^e_* d$). Choose $0 \neq c \in R$ and consider the map $\Phi_f : \operatorname{Hom}_R(F^f_* R, R) \xrightarrow{eval@c} R$ for some $f \gg 0$. Because $R[d^{-1}]$ is strongly F-regular, $d^m \in \operatorname{Image}(\Phi_f)$ for some $f \gg 0$ and some $m > 0$. Without loss of generality, we may assume that $m = p^l$ for some integer l (note making m larger is harmless). In particular, there exists $\psi \in \operatorname{Hom}_R(F^f_* R, R)$ such that $\psi(F^f_* c) = d^{p^l}$. Let $\kappa : F^l_* R \longrightarrow R$ be a Frobenius splitting and notice that $\kappa(F^l_* \psi(F^f_* c)) = \kappa(d^{p^l}) = d$. Finally

$$\phi(F^e_* \kappa(F^l_* \psi(F^f_* c))) = \phi(F^e_* d) = 1$$

and so $\phi \circ (F^e_* \kappa) \circ (F^{e+f}_* \psi)$ is the desired map. $\square$

For a computer, the above is not so bad. To show that $R = S/I$ (where S is a polynomial ring say) is strongly F-regular, you just need to find $d \in S$ in the ideal of the singular locus of $V(I)$ such that $\Phi(c \cdot (I^{[p^e]} : I)) = S$ where Φ is the map from the second Macaulay2 assignment. Note that this can only prove that a singularity is strongly F-regular, it can't prove that a singularity isn't. We don't have a good algorithm to do this in general but we do have algorithms that work if the ring is quasi-Gorenstein (or $\mathbb{Q}$-Gorenstein, a notion we'll learn about later).

1. A crash course in (ab)normality

Definition 1.1. Suppose that R is a ring. We let

$$K(R) = \{a/b \mid a, b \in R \text{ where } b \text{ is not a zero divisor } \}.$$

denote the *total ring of fractions.*

In the case that R is a reduced Noetherian ring with minimal primes $Q_1, \dots, Q_t$, then each R/Q_i is an integral domain with field of fractions $K(R/Q_i)$. In this case, $K(R) \cong \prod K(R/Q_i)$ (easy exercise, or look it up).

Definition 1.2. Given a reduced Noetherian ring R, the *normalization* R^{N} *of* R *in* $K(R)$ (or just the *normalization of* R) is defined to be

$$\{x \in K(R) \mid x \text{ satisfies a monic polynomial with coefficients in } R\}.$$

R is called *normal* if $R = R^{\mathrm{N}}$.

Fact 1.3. Under moderate hypotheses (excellence, so for all rings we care about), R^{N} is a finitely generated R-module. We will take this as a fact at least for now.

Lemma 1.4. *In a reduced ring R, the set of zero divisors is equal to $\bigcup Q_i$ where Q_i is the set of minimal primes.*

Proof. Suppose x is a zero divisor $xy = 0$, $y \neq 0$. If $x \notin Q_i$ for any i, then since $xy = 0 \in Q_i$, $y \in Q_i$ for all i. But $\bigcap Q_i = \langle 0 \rangle$ since R is reduced.

For the reverse direction fix some minimal prime Q_i and let W be the multiplicative set generated by $R \setminus Q_i$ and by the set of nonzero divisors of R. Note $0 \notin W$ because if it was, then $0 = ab$ for $a \notin Q_i$ and b not a zero divisor. Let $W^{-1}P$ be a maximal ideal of $W^{-1}R$ with $P \subseteq R$ the inverse image in R. Thus P is a prime ideal of R which doesn't contain any element of W. Obviously then $P \subseteq Q_i$, but since Q_i is minimal $P = Q_i$. But P doesn't contain any non-zero divisors, and so Q_i is completely composed of zero divisors. $\square$

Our goal for this section is to understand normal rings, non-normal rings, and some weakenings of the condition that R is normal. First let's understand $K(R)$.

Lemma 1.5. *Suppose that R is a reduced Noetherian ring, then $K(R) = \prod_{i=1}^t K_i$ is a finite product of fields.*

Proof. First we observe that R has only finitely many minimal primes. To see this write $\langle 0 \rangle = \bigcap_{i=1}^t P_i$ as a primary decomposition of $\langle 0 \rangle$. Any prime (minimal with respect to the condition that it contains 0 – any prime) is among this set (since the primes in primary decomposition commute with localization in as much as possible). Next let Q_i be the minimal primes, we claim that $\langle 0 \rangle = \bigcap_i Q_i$, one containment is obvious. On the other hand, if x is in every minimal prime then it is in every prime, and so x is nilpotent.

Now, if we localize a reduced ring at a minimal prime, we get a reduced ring with a single prime, in other words a field. Consider the diagonal map

$$\delta : R \longrightarrow \prod_i R_{Q_i}.$$

Note that $Q_i R_{Q_i}$ is zero since it's a nonzero ideal in a field hence each $R \longrightarrow R_{Q_i}$ factors through R/Q_i (which injects into R_{Q_i}). Thus $\ker \delta = \bigcap Q_i = \langle 0 \rangle$. On the other hand, every nonzero divisor of R certainly maps to a nonzero divisor of $\prod_i R_{Q_i}$ where it is already invertible and so we have map $\gamma : K(R) \longrightarrow \prod_i R_{Q_i}$. We need to show that this map is a bijection. It is injective since $K(R)$ is itself Noetherian and the minimal primes of R contain only zero divisors and so the minimal primes of $K(R)$ agree with the minimal primes of R. From hear on, we may assume $K(R) = R$. Let Q_1, Q_2 be minimal primes of $K(R)$ and consider $Q_1 + Q_2$. Since $Q_1 + Q_2$ is not contained in any single minimal prime Q_i, $Q_1 + Q_2$ is not contained in $\bigcup Q_i$ by prime avoidance. But in a reduced ring, $\bigcup Q_i$ is the set of zero divisors and so $Q_1 + Q_2$ contains a nonzero divisor and so $Q_1 + Q_2 = K(R)$ (since nonzero divisors are invertible). But now we've show that the Q_i are pairwise relatively prime and so γ is surjective by the Chinese Remainder Theorem. □

Lemma 1.6. *If we have an extension of rings $R \subseteq R' \subseteq K(R)$ such that R' is a finite R-module and R is Noetherian, then $R' = R$.*

Proof. Choose $x \in R'$ and so reduce to the case where $R' = R[x] \subseteq K(R)$. On the other hand, consider the ascending chain of R-submodules of $K(R)$ $R \subseteq R \oplus xR \subseteq R \oplus xR \oplus x^2 R \subseteq \cdots \subseteq \bigoplus_{i=1}^n x^i R \subseteq \ldots$. Eventually this stabilizes to R' and since R' is a Noetherian R-module, this happens at a finite step. Thus for some $n \gg 0$, $x^n \in \bigoplus_{i=1}^n x^i R \subseteq R'$. In other words, x satisfies a monic polynomial with coefficients in R and so $x \in R$ and thus $R' = R$ as claimed. □

Exercise 1.1. The formation of R^{N} commutes with localization, in particular if $W \subseteq R$ is a multiplicative set then $(W^{-1}R)^{\mathrm{N}} = W^{-1}(R^{\mathrm{N}})$.

As a corollary of the previous exercise, we immediately obtain the following.

Corollary 1.7. *A ring is normal if and only if each of its localizations $R_{\mathfrak{m}}$ are normal for maximal ideals $\mathfrak{m}$.*

Proposition 1.8. *If $(R, \mathfrak{m})$ is a reduced Noetherian local normal ring then R is an integral domain.*

Proof. Suppose that $K(R) = \prod_{i=1}^t K_i$ is a product of fields. We need to show that $t = 1$ since $R \subseteq K(R)$. □

Definition 1.9. A ring is called $\mathbf{R}_n$ if for every prime $Q \in \operatorname{Spec} R$ of height $\leq n$, R_Q is regular.

Lemma 1.10. *A Noetherian normal local 1-dimensional domain is regular. In particular, normal rings are $\mathbf{R}_1$.*

Proof. For the first statement, such a domain obviously has two prime ideals, 0 and $\mathfrak{m}$. We need to show that $\mathfrak{m}$ is principal and so choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Now, $R/\langle x \rangle$ is dimension zero and so as an R-module, has $\mathfrak{m}$ as an associated prime. Thus there exists $y \in R$ with $\overline{y} \in R/\langle x \rangle$ such that $\operatorname{Ann}_R \overline{y} = \mathfrak{m}$. In other words

$$y \notin \langle x \rangle \text{ but } y \cdot \mathfrak{m} \in \langle x \rangle.$$

Now consider $y/x \in K(R)$, we observe that $(y/x) \cdot \mathfrak{m} \subseteq R$ even though $y/x \notin R$ (since otherwise $y \in \langle x \rangle$).

Form $\mathfrak{m}^{-1} = R :_{K(R)} \mathfrak{m} = \{a \in K(R) \mid a\mathfrak{m} \subseteq R\}$ and consider $\mathfrak{m} \cdot \mathfrak{m}^{-1} \subseteq R$. Since $R \subseteq \mathfrak{m}^{-1}$, we see that $\mathfrak{m} \subseteq \mathfrak{m} \cdot \mathfrak{m}^{-1}$. By construction, $y/x \in \mathfrak{m}^{-1}$ and so if $\mathfrak{m} = \mathfrak{m} \cdot \mathfrak{m}^{-1}$, then $y/x \cdot \mathfrak{m} \subseteq \mathfrak{m}$. Thus we can view $(\cdot(y/x)) \in \operatorname{Hom}_R(\mathfrak{m}, \mathfrak{m})$ and thus $(y/x)^n + a_1(y/x)^{n-1} + \cdots + a_n = 0$ with $a_i \in \mathfrak{m}^i$ by the determinant trick (which leads to the proof of Nakayama's lemma). But then y/x is integral over R and thus since R is integrally closed, $y/x \in R$ a contradiction to the assumption that $\mathfrak{m} = \mathfrak{m} \cdot \mathfrak{m}^{-1}$. Thus $\mathfrak{m} \cdot \mathfrak{m}^{-1} = R$. Now consider $x \cdot \mathfrak{m}^{-1} \subseteq \mathfrak{m} \cdot \mathfrak{m}^{-1} = R$ and observe that if $x \cdot \mathfrak{m}^{-1} \subseteq \mathfrak{m}$, then $\langle x \rangle = x \cdot \mathfrak{m}^{-1}\mathfrak{m} \subseteq \mathfrak{m}^2$ contradicting our choice of x. Hence $x \cdot \mathfrak{m}^{-1} = R$ as well and so

$$\langle x \rangle = x \cdot \mathfrak{m}^{-1} \cdot \mathfrak{m} = \mathfrak{m} \cdot R = \mathfrak{m}$$

proving that $\mathfrak{m}$ is principal as desired.

The second statement is a direct corollary for the first since normality of Noetherian rings localizes by Exercise 1.1. □

Lemma 1.11. *A normal Noetherian reduced local ring $(R, \mathfrak{m})$ is a domain.*

Proof. Let $Q_1, \dots, Q_m$ be the minimal primes of R. Then consider the inclusion $R \hookrightarrow \prod_i R/Q_i = R'$. Obviously R' is a finite R-module (since it's a product of finitely many finite R-modules). Thus since R is normal, and obviously $R' \subseteq K(R)$, we have that $R = R'$. But R' is not local unless there is only one Q_i (each R/Q_i is local). □

Lemma 1.12. *A normal Noetherian ring with a dualizing complex is $\mathbf{S}_2$.*

Proof. The statement is local so we assume that R is a normal local domain. It is easy to see that normal domains are $\mathbf{S}_1$ since they are domains and so regular elements are not hard to find. Thus we need to show that $H^1_Q(R_Q) = 0$ for all $Q \in \operatorname{Spec} R$ of height at least 2. Thus we may as well assume $(R, \mathfrak{m})$ is a local Noetherian, normal domain of dimension ≥ 2 and set $U = \operatorname{Spec} R \setminus \mathfrak{m}$ to be the punctured spectrum. We denote by $\omega_R^{\bullet}$ a normalized dualizing complex. By **??** it suffices to show that

$$R \longrightarrow R' := \Gamma(U, R)$$

is surjective. It is not difficult to see that $R' \subseteq K(R)$ since if you tensor map defining R' by $K(R)$, the kernel of the tensored map is clearly $K(R)$. We know that the cokernel of $R \longrightarrow R'$ is $H^1_{\mathfrak{m}}(R)$. Since R is reduced it is $\mathbf{S}_1$ so it is an easy exercise in localizing dualizing complexes to verify that $h^{-1}\omega_R^{\bullet}$ has zero dimensional support (is supported at the closed point). In particular, $\mathfrak{m}^d \cdot h^{-1}\omega_R^{\bullet} = 0$ and so $h^{-1}\omega_R^{\bullet}$ is Artinian. It follows that its Matlis dual $H^1_{\mathfrak{m}}(R)$ is Noetherian. Thus R' is an extension of Noetherian R-modules, R and $H^1_{\mathfrak{m}}(R)$ and so R' is Noetherian. But now every element of R' is integral[1] over R (basically for the same reason that finite field extensions are algebraic) and so $R' \subseteq R^{\mathrm{N}} = R$ which completes the proof. □

[1]satisfy a monic polynomial equation

Theorem 1.13. *An excellent*[2] *reduced Noetherian ring R with a dualizing complex is normal if and only if it is $\mathbf{S}_2$ and $\mathbf{R}_1$.*

Proof. We already have seen that normal rings are $\mathbf{S}_2$ and $\mathbf{R}_1$. Conversely, if R is $\mathbf{S}_2$ and $\mathbf{R}_1$ and R^{N} is the normalization of R, then since R is is excellent, R^{N} is a finite R-module and so the locus where R is not normal, $Z = V(\mathrm{Ann}_R(R^{\mathrm{N}}/R))$, is closed. Since R is $\mathbf{R}_1$, Z has codimension at least 2 locally in $\operatorname{Spec} R$, thus $U = \operatorname{Spec} R \setminus Z$ is the complement of a codimension 2 set and so $R \longrightarrow \Gamma(U, R)$ is an isomorphism since R is $\mathbf{S}_2$ using **??**.

Claim 1.14. *R^{N} is $\mathbf{S}_2$ as an R-module.*

Proof of claim. Choose P a prime ideal of R and suppose that $P = Q \cap R$ a prime ideal of R^{N}. We know that $H^i_{QR^{\mathrm{N}}_Q}(R^{\mathrm{N}}) = 0$ for $i = 0, 1$ since R^{N} is $\mathbf{S}_2$. On the other hand, $\sqrt{PR^{\mathrm{N}}}$ is an intersection of finitely many prime ideals, $Q_1, \dots, Q_d$ such that $Q_j \cap R = P$. Then

$$H^i_{PR_P}(R^{\mathrm{N}}_P) = H^i_{PR^{\mathrm{N}}_P}(R^{\mathrm{N}}_P) = H^i_{\sqrt{PR^{\mathrm{N}}_P}}(R^{\mathrm{N}}_P) = \bigoplus_j H^i_{Q_j}(R^{\mathrm{N}}_{Q_j})$$

where the last equality comes from the fact that the functors $\Gamma_{\sqrt{PR^{\mathrm{N}}}}(\_\_) = \bigoplus_j \Gamma_{Q_j}(\_\_)$ for the semi-local ring R^{N}_P. But now we are done since the right side is zero for $i = 0, 1$. $\square$

Since R^{N} is a $\mathbf{S}_2$ R-module, we have that $R^{\mathrm{N}} \longrightarrow \Gamma(U, R^{\mathrm{N}})$ is an isomorphism. Finally, we see that $\Gamma(U, R) \longrightarrow \Gamma(U, R^{\mathrm{N}})$ is an isomorphism as well (since R and R^{N} agree on U). Putting this together we get the commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & R^{\mathrm{N}} \\ \sim\downarrow & & \downarrow\sim \\ \Gamma(U, R) & \xrightarrow[\sim]{} & \Gamma(U, R^{\mathrm{N}}) \end{array}$$

from which it follows that $R \longrightarrow R^{\mathrm{N}}$ is an isomorphism as well. $\square$

References

[2]We only include this to guarantee that R^{N} is a finitely generated R-module