NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 31ST, 2017

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1. Divisors, Frobenius splittings and finite extensions continued

Remember we are thinking about the following question. Suppose that $R \subseteq S$ is a finite extension of normal Noetherian domains. Suppose we have $\phi : F^e_* R \longrightarrow R$. It is natural to ask when ϕ extends to $\phi_S : F^e_* S \longrightarrow S$ and when it does, what is the relation between the divisors Δ_R and Δ_S.

Let us first analyze this in the case where R and S are fields.

Lemma 1.1. *Suppose K is an F-finite field and $\phi : F^e_* K = K^{1/p^e} \longrightarrow K$ is a nonzero map. If $K \subseteq L$ is a finite separable extension of fields then ϕ extends uniquely to a map $\phi_L : F^e_* L = L^{1/p^e} \longrightarrow L$.*

Proof. Since L and K^{1/p^e} are linearly disjoint extensions of K (one separable, the other purely inseparable), $L \otimes_K K^{1/p^e} = L \cdot K^{1/p^e} = L^{1/p^e}$ (to see the second equality, note we have the containment $\subseteq$ and also that $[L^{1/p^e} : L] = [K^{1/p^e} : K]$ since $[L^{1/p^e} : K^{1/p^e}] = [L : K]$). It follows that

$$\phi \otimes_K L : L^{1/p^e} \cong K^{1/p^e} \otimes_K L \longrightarrow K \otimes_K L$$

is a map extending ϕ. $\square$

Exercise 1.1. Suppose that L/K is a finite extension of characteristic $p > 0$ fields and $x \in L \setminus K$ but $x^p \in K$. Show that if $\phi : K^{1/p^e} \longrightarrow K$ extends to $L^{1/p^e} \longrightarrow L$, then ϕ is the zero map on K. Conclude in general that if L/K is inseparable, then no nonzero $\phi : K^{1/p^e} \longrightarrow K$ can extend to $L^{1/p^e} \longrightarrow L$.

Recall that given a finite extension of fields $K \subseteq L$, we have the trace map $\mathrm{Tr} : L \longrightarrow K$. This is defined as follows, for each $y \in L$, we have a K-linear map $L \xrightarrow{\cdot y} L$, $\mathrm{Tr}(y)$ is then defined to be the trace of the linear operator $\cdot y$.

Lemma 1.2. *Suppose L/K is a separable extension of fields and that $\phi : K^{1/p^e} \longrightarrow K$ extends to $\phi_L : L^{1/p^e} \longrightarrow L$. Let $\mathrm{Tr} : L \longrightarrow K$ be the trace map. Then the following diagram commutes*

$$\begin{array}{ccc} L^{1/p^e} & \xrightarrow{\phi_L} & L \\ {\scriptstyle \mathrm{Tr}^{1/p^e}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ K^{1/p^e} & \xrightarrow[\phi]{} & K. \end{array}$$

Proof. Choose $x_1^{1/p^e}, \ldots, x_n^{1/p^e}$ a basis for $K^{1/p^e}/K$. It follows since $L^{1/p^e} = L \cdot K^{1/p^e} = L \otimes_K K^{1/p^e}$ that $x_1^{1/p^e}, \ldots, x_n^{1/p^e}$ is a basis for $L^{1/p^e}/L$. On the other hand, if $z_1, \ldots, z_d$ is a basis for L/K, then it remains a basis for $L^{1/p^e}/K^{1/p^e}$. It follows that $(\mathrm{Tr})^{1/p^e}|_L = \mathrm{Tr}$. Now since ϕ extends to ϕ_L, $\phi(x_i^{1/p^e}) = \phi_L(x_i^{1/p^e})$. Finally, for any $y \in L$, write

$$y^{1/p^e} = \sum a_i x_i^{1/p^e}$$

for some $a_i \in L$. Thus

$$\begin{aligned} & \phi \circ \mathrm{Tr}^{1/p^e}(y^{1/p^e}) \\ = & \phi\big(\sum x_i^{1/p^e}(\mathrm{Tr}^{1/p^e}(a_i))\big) \\ = & \sum \phi(x_i^{1/p^e}) \,\mathrm{Tr}(a_i). \end{aligned}$$

and also

$$\begin{aligned} & \mathrm{Tr} \circ \phi_L(y^{1/p^e}) \\ = & \mathrm{Tr}(\sum a_i \phi(x_i^{1/p^e})) \\ = & \sum \mathrm{Tr}(a_i)\phi(x_i^{1/p^e}). \end{aligned}$$

The result follows. □

For the rest of the class, we did a worksheet on reflexive sheaves and divisors.

References