NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 6TH, 2017

KARL SCHWEDE

Last time we saw the following:

Theorem 0.1. *An excellent*[1] *reduced Noetherian ring R with a dualizing complex is normal if and only if it is $\mathbf{S}_2$ and $\mathbf{R}_1$.*

Remark 0.2. We included some hypotheses above to make our lives easier that are not strictly necessary. Indeed, the above holds even without assuming that R is excellent with a dualizing complex.

Note, in many cases it is easy to verify that certain rings are normal. Indeed if R is Gorenstein (for example, if it is defined by a hypersurface or a complete intersection) and $\mathbf{R}_1$, then it is automatically normal. For example, $k[x,y,z]/\langle x^a + y^b - z^c \rangle$ is normal if p does not divide a, b, c since then the singular locus is at the origin (using the Jacobian criterion).

In the previous proof, the ideal $\mathrm{Ann}_R(R^{\mathrm{N}}/R)$ appeared, this ideal has a special name.

Definition 0.3. The ideal $\mathfrak{c} := \mathrm{Ann}_R(R^{\mathrm{N}}/R)$ is called the *conductor of R^{N} over R*. It is also an ideal of R^{N}.

Let's now move towards a proof that strongly F-regular rings are normal.

Lemma 0.4. *Suppose that R is an F-finite Noetherian reduced ring of characteristic $p > 0$ and pick $\phi \in \mathrm{Hom}_R(F^e_* R, R)$. Then $\phi(F^e_* \mathfrak{c}) \subseteq \mathfrak{c}$.*

Proof. Tensoring $\phi : F^e_* R \longrightarrow R$ by $K(R)$ induces a map $\phi_{K(R)} : F^e_* K(R) \longrightarrow K(R)$ which restricts to ϕ and so which we also denote by ϕ. Choose $x \in \mathfrak{c}$ and $r \in R^{\mathrm{N}}$. Then $\phi(F^e_* x) \cdot r = \phi(F^e_*(r^{p^e} x))$. But $r^{p^e} \in R^{\mathrm{N}}$ and $x R^{\mathrm{N}} \subseteq R$ so $\phi(F^e_*(r^{p^e} x)) \in R$. Thus $\phi(F^e_* x) \cdot R^{\mathrm{N}} \subseteq R$ and so $\phi(F^e_* x) \in \mathfrak{c}$ as desired. □

Corollary 0.5. *Strongly F-regular rings are normal.*

Proof. If R was not normal, then the conductor satisfies $0 \neq \mathfrak{c} \neq R$ and so choose $0 \neq c \in \mathfrak{c}$. By hypothesis, $\phi(F^e_* c) \subseteq \mathfrak{c}$ and so $\phi(F^e_* c) \neq 1$ for any $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ which proves that R is not strongly F-regular. □

Example 0.6. $R = k[x,y]/\langle xy \rangle$ is F-split, by Fedder's criterion, but not normal since it is not $\mathbf{R}_1$. So we cannot weaken the above strongly F-regular hypothesis to simply being F-split.

Let's discuss an example of a non-normal ring.

[1] We only include this to guarantee that R^{N} is a finitely generated R-module

Example 0.7 (The node)**.** Consider the ring $S = k[x]$ and consider the subring $R = \{f \in S \mid f(0) = f(1)\}$ which one can also view as the pullback of the diagram

$$S \xrightarrow{\alpha} S/\langle x\rangle \cap \langle x-1 \xleftarrow{\beta} \leftarrow k.$$

In other words $\{(s,t) \in S \oplus k \mid \alpha(s) = \beta(t)\}$. We claim that

$$R = k[x(x-1), x^2(x-1)] \subseteq k[x] = S.$$

Obviously $x(x-1)$ and $x^2(x-1)$ are both in R. On the other hand, if $f \in R$ with $f(0) = f(1) = \lambda$, then $f - \lambda \in R$ and viewing $f - \lambda \in S$, we see that $f \in \langle x(x-1)\rangle_S = I_S \subseteq R$. Thus the question is, are the elements $a = x(x-1)$, $b = x^2(x-1)$ enough to produce all of I_S by multiplying a, b together and scaling them by elements of k. For example, if we have $hx(x-1) \in I$ with $h = h_0 + h_1x + h_2x^2 + \dots$, we can certainly assume that $h = h_2x^2 + \dots$ (since lower degree terms are easy to handle with a, b). But

$$c = x^2 \cdot x(x-1) = a^2 + b, d = x^3 * (x-1) = a * b - c$$

and so on...

On the other hand, it is easy to verify that $R = k[a,b]/\langle a^3 + a * b - b^2\rangle$.

Example 0.8. Similarly, it is not difficult to see that $R = k[x^2, x^3] \subseteq k[x] = S$ is the pullback of $(S \longrightarrow S/\langle x^2\rangle \leftarrow k)$. In particular, the cusp can be thought of as what you get when you kill first order tangent information at the origin of $\mathbb{A}^1$.

Theorem 0.9. *Suppose that we have a diagram of rings* $(A \twoheadrightarrow A/I \xleftarrow{g} B)$ *and let* $C = \{(a,b) \mid \overline{a} = g(b)\}$ *be the pullback. Then* $\operatorname{Spec} C$ *is the pushout of the diagram of induced map of topological spaces* $\{\operatorname{Spec} A \leftarrow \operatorname{Spec} A/I \rightarrow \operatorname{Spec} B\}$.

References