NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA
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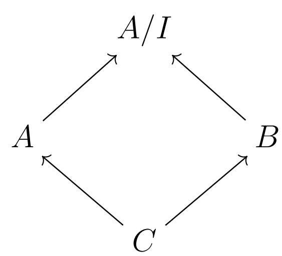
Theorem 0.1. *Suppose that we have a diagram of rings* $(A \twoheadrightarrow A/I \xleftarrow{g} B)$ *and let* $C = \{(a,b) \mid \overline{a} = g(b)\}$ *be the pullback. Then* $\operatorname{Spec} C$ *is the pushout of the diagram of induced map of topological spaces* $\{\operatorname{Spec} A \leftarrow \operatorname{Spec} A/I \rightarrow \operatorname{Spec} B\}$.

Proof. I will only sketch this in the category of sets, I will leave the verification that it behaves properly on thhe level of topological spaces as an exercise to the reader.

First consider the following ideal $J = \{(i,0) \mid i \in I\} \subseteq C$. It is easy to see that $C/J \cong B$ and so C contains a copy of $\operatorname{Spec} C$, $V(J) \subseteq \operatorname{Spec} C$. On the other hand, for any prime ideal of $\operatorname{Spec} C$ not containing J, doesn't contain some $(i,0)$ and so if we let $W = \{(i,0),(i^2,0),\dots\}$ denote the induced multiplicative set, we see that $W^{-1}(C) \cong W^{-1}A = A[i^{-1}]$. From this it is not hard to see that the primes of $\operatorname{Spec} C$ that don't contain J correspond precisely to the primes of $\operatorname{Spec} A$ that don't contain I (the map $\operatorname{Spec} C \longrightarrow \operatorname{Spec} A$ is an isomorphism outside of $V(J)$ and $V(I)$). Putting this together plus the commutative diagram



is enough to prove the result. □

We now prove that every non-normal ring arises this way.

Proposition 0.2. *Suppose that* R *is a reduced non-normal ring with normalization* R^{N} *and suppose that* $R \subseteq S \subseteq R^{\mathrm{N}}$. *Let* $\mathfrak{c} = \operatorname{Ann}_R(S/R)$ *denote the conductor of* $R \subseteq S$ *and recall it is an ideal in both* R *and* S. *The* R *is the pullback of* $(S \longrightarrow S/\mathfrak{c} \leftarrow R/\mathfrak{c})$.

Proposition 0.3. *Let* C *denote the pullback and so by the universal property, we have a map* $R \longrightarrow C$. *We need to prove it is an isomorphism. It is obviously an injection since we already have* $R \subseteq S$. *Thus we need to show that* $R \longrightarrow C$ *is surjective. Choose an element* $(s, \overline{r}) \in C$. *Choose* $r \in R$ *whose image in* $R/\mathfrak{c}$ *is* $\overline{r}$. *Now then* $s - r \in R^{\mathrm{N}}$ *and is sent to zero* $S/\mathfrak{c}$ *and so* $s - r \in \mathfrak{c} \subseteq R$. *Therefore* $s \in R$ *as well and so* $(s, \overline{r})$ *is the image of* $s \in R$ *which proves that* $R \longrightarrow C$ *is an isomorphism.*

In other words, a normal ring is a ring without any excess gluing and non-normal rings are obtained from normal ones by gluing points (or subschemes) together and killing (higher) tangent spaces. At this point it is also not hard understand where non-$\mathbf{S}_2$ rings come from as well. They arise as gluings where the conductor has information in codimension 2. On the other hand if your gluing information is pure codimension 1, then the resulting non-normal ring will be $\mathbf{S}_2$ but not $\mathbf{R}_1$.

The rest of the time was spent on the worksheet on implications of Serre's conditions.

References