NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA APRIL 10TH, 2017

KARL SCHWEDE

1. Hilbert-Kunz multiplicity

We continue our study of Hilbert-Kunz multiplicity.

1.1. **Existence of the limit.** We follow closely the recent proof of the existence of Hilbert-Kunz multiplicity as shown in [PT16].

First we state a fact that we won't have time to prove.

Lemma 1.1. *If $(R, \mathfrak{m}, k)$ is a local F-finite Noetherian domain of dimension d then*

$$[F^e_* K(R) : K(R)] = [F^e_* k : k] \cdot p^{ed}.$$

You probably already believe it anyways.

Lemma 1.2. *Suppose that $(R, \mathfrak{m}, k)$ is a local ring and that M is a finite R-module. Then there exists a constant (depending on M) so that*

$$\ell_R\big(M/\mathfrak{m}^{[p^e]}M\big) \leq C \cdot p^{e \dim M}.$$

Proof. Suppose that $\mathfrak{m}$ is generated by t elements, then $\mathfrak{m}^{tp^e} \subseteq \mathfrak{m}^{tp^e-t+1} \subseteq \mathfrak{m}^{[p^e]}$ by the pigeon-hole principal. But

$$\ell_R\big(M/\mathfrak{m}^{tp^e}\big)$$

is eventually a polynomial

$$D \cdot (tp^e)^{\dim M} + \dots = D \cdot t^{\dim M} \cdot p^{e \dim M} + \dots$$

of degree degree $\dim M$ in p^e. We can thus pick $C = D \cdot t^{\dim M}$ for $e \gg 0$, and choosing C even bigger for finitely many smaller e completes the proof. $\square$

Remark 1.3. The fact that $\ell_R\big(M/\mathfrak{m}^{tp^e}\big)$ is eventually a polynomial can be found for example in [AM69, Chapter 11] for the case that $M = R$. For the general case, if you only want to bound its length by a polynomial of degree $\dim M$ (which is all we actually need), write $J = \operatorname{Ann}_R(M)$ and note that $\dim R/J = \dim M$ and that there exists a surjection $(R/J)^{\oplus b} \longrightarrow M \longrightarrow 0$ for some $b > 0$.

Lemma 1.4. *Let R be an F-finite domain with $[F_* K(R) : K(R)] = p^\gamma$. Then there exists a short exact sequence*

$$0 \longrightarrow R^{\oplus p^\gamma} \longrightarrow F_* R \longrightarrow M \longrightarrow 0$$

such that $\dim(M) < \dim(R)$.

Proof. After inverting an element $c \in R$, we have that $R_c^{\oplus p^\gamma} \cong F_* R_c$. This gives us an injective map $R^{\oplus p^\gamma} \longrightarrow F_* R_c$. If the image is not in $F_* R \subseteq F_* R_c$, then multiplying by a high power of $c \in R$ will make it in R. This gives us the first map. Then if we we let M be the cokernel, it is easy to see that $\operatorname{Supp} M \subseteq V(c)$ and so $\dim(M) \leq \dim(R/cR) < \dim R$. $\square$

Theorem 1.5. *Suppose that $(R, \mathfrak{m}, k)$ is an F-finite Noetherian local domain of characteristic $p > 0$ and dimension $d > 0$ and that $\{I_e\}$ is a sequence of ideals such that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for all $e > 0$. Suppose further that $I_e^{[p]} \subseteq I_{e+1}$. Then*

$$\lim_{e \longrightarrow \infty} \frac{\ell_R(R/I_e)}{p^{ed}}$$

exists

Proof. We follow [PT16]. Consider a short exact sequence as in Lemma 1.4

$$0 \longrightarrow R^{\oplus p^\gamma} \xrightarrow{\phi} F_* R \longrightarrow M \longrightarrow 0$$

with $\dim M < \dim R$. Now, $I_e^{[p]} \subseteq (I_{e+1})$ implies that $I_e F_* R \subseteq F_* I_{e+1}$ and so

$$\phi(I_e^{\oplus p^\gamma}) = \phi(I_e \cdot R^{\oplus p^\gamma}) \subseteq I_e F_* R = F_* I_e^{[p]} \subseteq F_* I_{e+1}.$$

Thus we have

$$\overline{\phi} : (R/I_e)^{p^\gamma} \longrightarrow F_*(R/I_{e+1}).$$

Therefore the length of $(R/I_e)^{p^\gamma}$ plus the length of the cokernel of ϕ is at least the length of $F_*(R/I_{e+1})$. In other words

$$\ell_R\big(\operatorname{coker} \overline{\phi}\big) + p^\gamma \ell_R(R/I_e) \geq \ell_R(F_* R/I_{e+1}) = [k^{1/p} : k] \cdot \ell_R(R/I_{e+1}). \tag{1.5.1}$$

Now, $\operatorname{coker} \overline{\phi}$ is the image of $F_*(R/I_{e+1})$ and so $F_* I_{e+1}$ annihilates it. But

$$\mathfrak{m}^{[p^e]} \subseteq F_* \mathfrak{m}^{[p^{e+1}]} \subseteq F_* I_{e+1}$$

and so $\mathfrak{m}^{[p^e]}$ annihilates $\operatorname{coker} \overline{\phi}$. But $\operatorname{coker} \overline{\phi}$ is also the image over M and hence of $M/\mathfrak{m}^{[p^e]} M$. In particular

$$\ell_R\big(\operatorname{coker} \overline{\phi}\big) \leq \ell_R\big(M/\mathfrak{m}^{[p^e]} M\big).$$

By Lemma 1.2, this is bounded by $C_M p^{e(d-1)}$. Now, we divide (1.5.1) by $[F_* k : k] p^{(e+1)d} = p^{ed+\gamma}$ and obtain

$$\frac{C_M/p^\gamma}{p^e} + \frac{1}{p^{ed}} \ell_R(R/I_e) \geq \frac{1}{p^{(e+1)d}} \ell_R(F_* R/I_{e+1}). \tag{1.5.2}$$

The existence of the limit now follows from the following lemma. □

Lemma 1.6. *Suppose that p is prime, $d > 0$ an integer, and $\{t_e\}$ is a sequence of real real numbers such that $\{t_e/p^{ed}\}$ is bounded. Further suppose that there exists a constant C such that*

$$C/p^e + \frac{1}{p^{ed}} t_e \geq \frac{1}{p^{(e+1)d}} t_{e+1}.$$

Then $t := \lim_{e \longrightarrow \infty} \frac{t_e}{p^{ed}}$ exists and $t - 1/p^{ed} t_e \leq \frac{2C}{p^e}$.

References

[AM69] M. F. Atiyah and I. G. Macdonald: *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR0242802 (39 #4129)

[PT16] T. Polstra and K. Tucker: *F-signature and hilbert-kunz multipicity: a combined approach and comparison*, arXiv:1608.02678.