NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA APRIL 12TH, 2017

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1. Hilbert-Kunz multiplicity

We continue our study of Hilbert-Kunz multiplicity.

1.1. **Existence of the limit.** Last time, in order to prove that Hilbert-Kunz multiplicity existed, we were proving the following theorem.

Theorem 1.1. *Suppose that $(R, \mathfrak{m}, k)$ is an F-finite Noetherian local domain of characteristic $p > 0$ and dimension $d > 0$ and that $\{I_e\}$ is a sequence of ideals such that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for all $e > 0$. Suppose further that $I_e^{[p]} \subseteq I_{e+1}$. Then*

$$\lim_{e \longrightarrow \infty} \frac{\ell_R(R/I_e)}{p^{ed}}$$

exists

We concluded by showing that it was enough to prove the following lemma, again taken from [PT16].

Lemma 1.2. *Suppose that p is prime, $d > 0$ an integer, and $\{t_e\}$ is a sequence of real real numbers such that $\{t_e/p^{ed}\}$ is bounded. Further suppose that there exists a constant C such that*

$$C/p^e + \frac{1}{p^{ed}} t_e \geq \frac{1}{p^{(e+1)d}} t_{e+1}.$$

Then $t := \lim_{e \longrightarrow \infty} \frac{t_e}{p^{ed}}$ exists and $t - 1/p^{ed} t_e \leq \frac{2C}{p^e}$.

Proof. Note that

$$C/p^{e+1} + C/p^e + \frac{1}{p^{ed}} t_e \geq C/p^{e+1} + \frac{1}{p^{(e+1)d}} t_{e+1} \geq \frac{1}{p^{(e+2)d}} t_{e+2}$$

and more generally that

$$2C/p^e + \frac{1}{p^{ed}} t_e \geq C(1/p^{e+m+1} + \cdots + 1/p^e) + \frac{1}{p^{ed}} t_e \geq \frac{1}{p^{(e+m)d}} t_{e+m}.$$

Let t^+ denote the limit supremum of $\{t_e/p^{ed}\}$ and t^- the limit infimum. Note

$$2C/p^e + \frac{1}{p^{ed}} t_e \geq t^+.$$

Apply the limit infimum to both sides we get $t^- \geq t^+$ and so the limit exists and hence so does the desired bound. $\square$

Theorem 1.3. *For any $\mathfrak{m}$-primary ideal in a domain, and $J \subseteq \mathfrak{m}$ is $\mathfrak{m}$-primary, $e_{\mathrm{HK}}(J; R)$ exists.*

Proof. Suppose first that R is an F-finite domain. We'd be tempted to show that $\mathfrak{m}^{[p^e]} \subseteq J^{[p^e]}$, but this is impossible unless $J = \mathfrak{m}$. However, we certainly have $\mathfrak{m}^{[p^t]} \subseteq J$ for some integer t and so $\mathfrak{m}^{[p^e]} \subseteq J^{[p^{e-t}]}$ for all e. Hence we see that

$$\lim_{e \longrightarrow \infty} \frac{1}{p^{ed}} \ell_R\Big(R/J^{[p^{e-t}]}\Big) = \frac{1}{p^{td}} \lim_{e \longrightarrow \infty} \frac{1}{p^{ed}} \ell_R\big(R/J^{[p^e]}\big)$$

exists (say it equals b). Thus so does

$$bp^{td} = \lim_{e \longrightarrow \infty} \frac{1}{p^{ed}} \ell_R\big(R/J^{[p^e]}\big) = e_{\mathrm{HK}}(J; R).$$

For the non-F-finite case, it is harmless to assume that $R = k[\![x_1, \ldots, x_m]\!]/I$ is complete with residue/coefficient field k since the lengths $R/J^{[p^e]}$ are unchanged by completion. But now $S = \widehat{R \otimes_k \overline{k}}$ is F-finite and the R-lengths of $R/J^{[p^e]}$ are equal to the S-lengths of $S/(J^{[p^e]}S)$. □

References

[PT16] T. Polstra and K. Tucker: *F-signature and hilbert-kunz multipicity: a combined approach and comparison*, arXiv:1608.02678.