NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA APRIL 14TH, 2017

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1. F-signature

Suppose that $(R, \mathfrak{m}, k)$ is an F-finite domain of dimension d. For now, suppose that $k = k^p$ is perfect. Write

$$F^e_* R = R^{\oplus a_e} \oplus M_e$$

where M_e has no free summands. It turns out (we'll see shortly) that even though this sort of decomposition is not necessarily unique (it is if R is complete), the number a_e is independent of the decomposition. We'll define

$$s(R) = \lim_{e \longrightarrow \infty} \frac{a_e}{p^{ed}}$$

to be the F-signature of R. Note

(i) If R is regular, then $s(R) = 1$.
(ii) If R is not F-split, then $s(R) = 0$ (in fact, we'll see that if R is not F-regular, then $s(R) = 0$).
(iii) In general $0 \leq s(R) \leq 1$.

1.1. **Existence of F-signature.** Let's define $I_e = \{r \in R \mid \phi(F^e_* r) \subseteq \mathfrak{m} \text{ for all } \phi \in \mathrm{Hom}_R(F^e_* R, R)\}$.

Lemma 1.1. *I_e is an ideal of R.*

Proof. Obviously I_e is closed under sum. If $x \in I_e$ and $r \in R$, we need to show that $rx \in R$. For each $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, define $\phi'(F^e_* \_\_) = \phi(F^e_* x \_\_)$. Then $\phi'(F^e_* r) \in \mathfrak{m}$ so then $\phi(F^e_* xr) \in \mathfrak{m}$ and so I_e is an ideal. $\square$

Lemma 1.2. *With notation as above, $\ell_R(R/I_e) = a_e$. In particular, a_e is independent of the decomposition.*

Proof. Since M_e has no free R-summands, $\phi(M_e) \subseteq \mathfrak{m}$ for all $\phi \in \mathrm{Hom}_R(F^e_* R, R)$. Thus $M_e \subseteq I_e$. In fact we even have:

$$\mathfrak{m}^{\oplus a_e} \oplus M_e \subseteq I_e.$$

On the other hand, if $F^e_* x \in (F^e_* R) \setminus (\mathfrak{m}^{\oplus a_e} \oplus M_e)$, say the ith term in the direct sum decomposition is a unit u not in $\mathfrak{m}$. But then the projection onto the ith term sends $F^e_* x \mapsto u \notin \mathfrak{m}$. Hence

$$\mathfrak{m}^{\oplus a_e} \oplus M_e = I_e.$$

But $\ell R R/I_e = a_e$ as desired. $\square$

Obviously $I_e \supseteq \mathfrak{m}^{[p^e]}$. Hence the F-signature limit exists if can show that

$$I_e^{[p]} \subseteq I_{e+1}$$

by our previous work with limits.

Recall the theorem we proved last time.

Theorem 1.3. *Suppose that $(R, \mathfrak{m}, k)$ is an F-finite Noetherian local domain of characteristic $p > 0$ and dimension $d > 0$ and that $\{I_e\}$ is a sequence of ideals such that $\mathfrak{m}^{[p^e]} \subseteq I_e$ for all $e > 0$. Suppose further that $I_e^{[p]} \subseteq I_{e+1}$. Then*

$$\lim_{e \longrightarrow \infty} \frac{\ell_R(R/I_e)}{p^{ed}}$$

exists

We use this to prove that F-signature exists.

Lemma 1.4. *With notation as above $I_e^{[p]} \subseteq I_{e+1}$ and hence the F-signature limit exists by Theorem 1.3.*

Proof. Choose $r \in I_e$. Then for any $\phi \in \operatorname{Hom}_R(F_*^{e+1}R, R)$, we have a restricted $\psi = \phi|_{F_*^e R}$. Note the elements in $F_*^e R \subseteq F_*^{e+1} R$ are the pth powers of elements of $F_*^{e+1}R$, hence

$$\phi(F_*^{e+1} r^p) = \psi(F_*^e r) \in \mathfrak{m}.$$

Thus $I_e^{[p]} \subseteq I_{e+1}$ as desired. □

Remark 1.5. One can define $s(R) = \lim_{e \longrightarrow \infty} \ell_R(R/I_e)$ even if the residue field is not perfect (but still assuming R is F-finite). Our work above shows that the limit still exists.

References