NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA APRIL 17TH, 2017

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1. F-signature continued

1.1. **Positivity of F-signature.** Our next goal is to explain when the F-signature is positive. It is obviously zero if R is not F-split. First we need a lemma.

Lemma 1.1. *With notation as in the beginning of the section $\bigcap_e I_e = 0$ if and only if R is strongly F-regular.*

Proof. Exercise! □

Theorem 1.2. *$s(R) > 0$ if and only if R is strongly F-regular.*

Proof. We suppose that the residue field is perfect for simplicity.

Suppose first that $\bigcap_e I_e \neq 0$ (ie, that R is not strongly F-regular) that $0 \neq c \in \bigcap_e I_e$. Since $\mathfrak{m}^{[p^e]} \subseteq I_e$, we see that

$$\ell_R(R/I_e) \leq \ell_R\left(\frac{R}{\langle c\rangle + \mathfrak{m}^{[p^e]}}\right) \leq Cp^{e(d-1)}.$$

Therefore

$$s(R) = \lim_{e\longrightarrow\infty} \frac{1}{p^{ed}}\ell_R(R/I_e) = 0.$$

Now assume that R is strongly F-regular. Without loss of generality we may assume that $(R, \mathfrak{m}, k)$ is complete. The Cohen-Gabber-Structure Theorem says that we can find $A = k[\![x_1, \dots, x_n]\!] \subseteq R$ a finite *separable* extension (Noether normalization for complete rings). Furthermore, we can choose $0 \neq c \in A$ such that

$$c \cdot R^{1/p^e} \subseteq R[A^{1/p^e}] \cong R \otimes_A A^{1/p^e}$$

for all e. In other words, $c \cdot F^e_* R \subseteq R[F^e_* A]$. Now, since R is strongly F-regular, we can find $\phi \in \mathrm{Hom}_R(R^{1/p^{e_c}}, R)$ for some $e_c > 0$ such $\phi(c^{1/p^{e_c}}) = 1$. We will show that $s(R) \geq 1/p^{e_c d} > 0$.

Now, the p^eth roots of the monomials $\mathbf{x}^\alpha$, $\mathbf{0} \leq \alpha \leq \mathbf{p^e - 1}$ for a basis for A^{1/p^e} over A. Let p_α be the projection so that $p_\alpha(\mathbf{x}^{\alpha/p^e}) = 1$ and $p_\beta(\mathbf{x}^{\beta/p^e}) = 0$ for $\beta \neq \alpha$. We form the compositions

$$\pi_\alpha : R^{1/p^e} \xrightarrow{\cdot c} R \otimes_A A^{1/p^e} \xrightarrow{R\otimes\pi_\alpha} R.$$

Note $\pi_\alpha(\mathbf{x}^{\alpha/p^e}) = p_\alpha(c\mathbf{x}^{\alpha/p^e}) = c$. Now we post compose with ϕ and we obtain $\phi_\alpha = \phi \circ (\pi_\alpha)^{1/p^{e_c}}$ which sends $\mathbf{x}^{\alpha/p^{e+e_c}} \mapsto 1$ and $\mathbf{x}^{\beta/p^{e+e_c}} \mapsto 0$ for $\beta \neq \alpha$. Taking the direct sum of these maps gives a surjection

$$(\oplus\phi_\alpha) : R^{1/p^{e+e_c}} \longrightarrow R^{\oplus p^{ed}}.$$

Hence $s(R) \geq 1/p^{e_c d}$. □

Question 1.3. If one starts in characteristic zero with $R_{\mathbb{C}}$, can one find a lower bound on F-signatures of the mod p reductions $s(R_p)$? Better yet, it would be better to find a geometric interpretation of

$$\lim_{p \longrightarrow \infty} s(R_p)$$

(say in the case that R is essentially of finite type over $\mathbb{Q}$).

We spent the rest of the class with student presentations.

References