NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA APRIL 19TH, 2017

KARL SCHWEDE

1. Transformation of F-signature under étale in codimension 1 extensions

Our goal in this section is to understand how F-signature behaves under finite extensions.

Lemma 1.1. *Suppose that $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, k)$ is a finite extension of local domains with the same residue field. Then $\mathrm{Tr}(\mathfrak{n}) \subseteq \mathfrak{m}$.*

Proof. We prove it only in the case where the extension is generically Galois since the proof there is easier.

Let $G = \mathrm{Gal}(L/K)$ where $L = K(S)$ and $K = K(R)$. Then $\mathrm{Tr}(x) = \sum_{g \in G} g.x$. But if $x \in \mathfrak{n}$, then $g.x \in g.\mathfrak{n} = \mathfrak{n}$ (since the extension is local and finite). Thus $\mathrm{Tr}(x) \in \mathfrak{n} \cap R = \mathfrak{m}$. □

Lemma 1.2. *Suppose that $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, k)$ is a finite extension of local domains with the same residue field. Further suppose that $R \subseteq S$ is étale in codimension 1 (or in other words that the ramification divisor is zero). Then S has at most one free R-summand.*

Proof. The fact that the ramification divisor is zero means that $\mathrm{Tr} \in \mathrm{Hom}_R(S, R)$ generates the Hom-set. Consider $J = \{s \in S \mid \mathrm{Tr}(sS) \subseteq \mathfrak{m}\}$. Now $\ell_S(S/J)$ is the number of free R-summands by the same argument as **??**. But this length is at most one by Lemma 1.1 since $\mathfrak{n} \subseteq J$. □

We need another result we probably should have proven earlier.

Theorem 1.3. *If R is strongly F-regular and $R \subseteq S$ is a module finite extension, then $R \longrightarrow S$ splits as a map of R-modules.*

Proof. For any $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, consider the composition

$$\eta : F^e_* \mathrm{Hom}_R(S, R) = \mathrm{Hom}_{F^e_* R}(F^e_* S, F^e_* R) \longrightarrow \mathrm{Hom}_R(S, F^e_* R) \longrightarrow \mathrm{Hom}_R(S, R)$$

where the first map is induced by restriction and the second by ϕ. It is not difficult to verify that the follow diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{F^e_* R}(F^e_* S, F^e_* R) & \xrightarrow{\ \eta\ } & \mathrm{Hom}_R(S, R) \\ \Big\downarrow{\scriptstyle \mathrm{ev}@1} & & \Big\downarrow{\scriptstyle \mathrm{ev}@1} \\ F^e_* R & \xrightarrow[\ \phi\ ]{} & R. \end{array}$$

But then the image of $\mathrm{Hom}_R(S, R)$ in R is stable under every ϕ. Any nonzero element in the image can be sent to 1 by some ϕ and hence $\mathrm{Hom}_R(S, R) \longrightarrow R$ must surject. □

We spent the rest of the class on presentations.

References