NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA APRIL 3RD, 2017

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1. Divisors, Frobenius splittings and finite extensions continued

Last time we proved the following Lemma.

Lemma 1.1. *Suppose L/K is a separable extension of fields and that $\phi : K^{1/p^e} \longrightarrow K$ extends to $\phi_L : L^{1/p^e} \longrightarrow L$. Let $\mathrm{Tr} : L \longrightarrow K$ be the trace map. Then the following diagram commutes*

$$\begin{array}{ccc} L^{1/p^e} & \xrightarrow{\phi_L} & L \\ {\scriptstyle \mathrm{Tr}^{1/p^e}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ K^{1/p^e} & \xrightarrow[\phi]{} & K. \end{array}$$

We need the following standard result because we are going to restrict the compatibility from last time to an extension of rings $R \subseteq S$.

Lemma 1.2. *Suppose $R \subseteq S$ is a finite*[1] *inclusion of normal Noetherian domains. Let $\mathrm{Tr} : K(S) \longrightarrow K(R)$ denote the trace map, then $\mathrm{Tr}(S) \subseteq R$.*

Proof. Since R is $\mathbf{S}_2$, it suffices to prove the result after localizing at a height one prime of R (to see this, simply notice that $\mathrm{Tr}(S) \subseteq R$ is a finite R-module and so if it agrees with R in codimension 1, it is contained in R). But since R is normal, such a localization is a DVR and so we may assume that R is a DVR. But now S is a free R-module so a basis for S/R becomes a basis for L/K. It follows that for any $s \in S$, $S \xrightarrow{\cdot s} S$ is written as a matrix with entries in R and so $\mathrm{Tr}(s) \in R$ as claimed. □

We now also explain how to pullback divisors under finite maps of normal domains.

Definition 1.3. Suppose $R \subseteq S$ is a finite inclusion of normal domains with induced map $\pi : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. For any Weil divisor D on R, we define $\pi^* D$ to be the divisor such that $(R(D) \cdot S)^{\vee\vee} = S(\pi^* D)$. Alternately, for each height one prime Q of R, let $Q_1, \dots, Q_d$ be the primes of S lying over Q. In this case $R(D)_Q = g_Q R_Q$ since each R_Q is a DVR. We define $\pi^* D = \sum_Q \sum_{Q_i} -v_{Q_i}(g_Q) Q_i$. In particular, if $D = \mathrm{div}_R(f)$, then $\pi^* D = \mathrm{div}_S(f)$.

Example 1.4. If $S = F^e_* R$, then $\pi^* D = p^e D$.

If $R \subseteq S$ is a finite inclusion of normal domains such that $K(R) \subseteq K(S)$ is separable (with $\pi : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ the induced map), the map $\mathrm{Tr} \in \mathrm{Hom}_R(S, R)$ is a nonzero[2]

[1] Finite means S is a finitely generated R-module.

[2] Tr of an extension of fields is nonzero if and only if the extension is separable.

element in a rank-1 $\mathbf{S}_2$ S-module. Also note that if $\omega_S := \mathrm{Hom}_S(R, \omega_S)$, then

$$\begin{array}{rl} & \mathrm{Hom}_R(S, R) \\ = & \mathrm{Hom}_R(S \otimes_R \omega_R, \omega_R) \\ = & \mathrm{Hom}_R(S(\pi^* K_R), \omega_R) \\ = & \mathrm{Hom}_S(S(\pi^* K_R), \omega_S) \\ = & S(K_S - \pi^* K_R). \end{array}$$

and so the effective divisor D_{Tr} corresponding to Tr is linearly equivalent to $K_S - \pi^* K_R$.

Definition 1.5. With notation as above the effective divisor D_{Tr} corresponding to Tr is called the *ramification divisor (of $R \subseteq S$)*. Throughout the rest of the paper, it will be denoted by $\mathrm{Ram} = \mathrm{Ram}_{S/R}$.

You may have seen the ramification divisor defined somewhat differently, these two definitions are indeed the same.

We spent the rest of the class thinking about the worksheet.

References