NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA MARCH 29TH, 2017

KARL SCHWEDE

1. Divisors, Frobenius splittings and finite extensions continued

Last time we saw that if $R \subseteq S$ is a finite inclusion of normal domains then $\mathrm{Hom}_R(S, R) \cong S(K_S - \pi^* K_R)$ (assuming ω_S is defined to be $\mathrm{Hom}_R(S, \omega_R)$). We then made the following definition.

Definition 1.1. With notation as above the effective divisor D_{Tr} corresponding to Tr is called the *ramification divisor (of $R \subseteq S$)*. Throughout the rest of the paper, it will be denoted by $\mathrm{Ram} = \mathrm{Ram}_{S/R}$.

You may have seen the ramification divisor defined somewhat differently, these two definitions are indeed the same. Let's do a hopefully convincing example.

Example 1.2. Consider $R = k[x^n] \subseteq k[x] = S$ where char$k$ does not divide n. We will compute the trace map and thus the ramification divisor. Note that S is a free R-module with basis $1, x, \ldots, x^{n-1}$. It is easy to see that $\mathrm{Tr}(x^i) = 0$ for $0 < i < n$ and $\mathrm{Tr}(1) = n$ (just write down the matrices). On the other hand the map Φ that projects onto x^{n-1} obviously generates $\mathrm{Hom}_R(S, R)$ as an S-module. Note $\mathrm{Tr}(\_\_) = n \cdot \Phi(x^{n-1}\_\_)$ so that because $D_\Phi = 0$, we see that $D_{\mathrm{Tr}} = \mathrm{div}(x^{n-1})$.

Definition-Proposition 1.3. If $R \subseteq S$ is a finite separable extension of rings with R a DVR with uniformizer s, then it is said to have *tame ramification* at a maximal ideal $Q \in \mathrm{Spec}\, S$ (so that S_Q is a DVR with uniformizer s) if it satisfies the following two conditions:

- when we write $r = us^n$, p does not divide n
- and if $R/rR \subseteq S_Q/sS_Q$ is separable.

In this case, the coefficient of Ram at Q is equal to $n - 1$.

Exercise 1.1. Verify the above definition - proposition.

Lemma 1.4. *Suppose we have finite inclusions of normal domains $A \subseteq B \subseteq C$ with $\rho : \mathrm{Spec}\, C \longrightarrow \mathrm{Spec}\, B$ the induced map. If $\phi \in \mathrm{Hom}_A(B, A)$ and $\psi \in \mathrm{Hom}_B(C, B)$ then $D_{\phi \circ \psi} = D_\psi + \rho^* D_\phi$.*

Proof. Since we are concerned above divisors, we may assume that A is a DVR so that B and C are semi-local. In this case if Φ and Ψ generate their respective Hom groups, we see from an older homework assignment that so does $\Phi \circ \Psi$ and thus all divisors in question are zero. On the other hand, if $\phi(\_\_) = \Phi(b \cdot \_\_)$ and $\psi(\_\_) = \Psi(c \cdot \_\_)$, then $D_\phi = \mathrm{div}_B(b)$ and $D_\psi = \mathrm{div}_C(c)$ and $\mathrm{div}_{\phi\circ\psi} = \mathrm{div}_C(bc) = D_\psi + \rho^* D_\phi$. $\square$

Theorem 1.5. *Suppose that $R \subseteq S$ is a finite generically separable inclusion of F-finite normal domains with induced $\pi : \mathrm{Spec}\, S \longrightarrow \mathrm{Spec}\, R$. Then a map $\phi \in \mathrm{Hom}_R(F^e_* R, R)$*

extends to a map $\phi_S \in \mathrm{Hom}_S(F^e_* S, S)$ *if and only if* $\pi^*\Delta_\phi - \mathrm{Ram} \geq 0$. *In this case* $\Delta_{\phi_S} = \pi^*\Delta_\phi - \mathrm{Ram}$.

Proof. Suppose first that ϕ extends to ϕ_S (note Δ_{ϕ_S} is automatically effective) and so by **??** we have a commutative diagram

$$\begin{array}{ccc} S^{1/p^e} & \xrightarrow{\phi_S} & S \\ {\scriptstyle \mathrm{Tr}^{1/p^e}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ R^{1/p^e} & \xrightarrow[\phi]{} & R. \end{array}$$

It follows from Lemma 1.4 that $\mathrm{Ram} + \pi^* D_\phi = D_{\phi_S} + p^e\mathrm{Ram}$. Thus $0 \leq \Delta_{\phi_S} = \frac{1}{p^e-1}(\pi^* D_\phi) - \mathrm{Ram} = \pi^*\Delta_\phi - \mathrm{Ram}$.

Conversely assume that $\pi^*\Delta_\phi - \mathrm{Ram} \geq 0$. We can still extend ϕ to $\phi_S : S^{1/p^e} \longrightarrow L = K(S)$. We need to show that the image of is contained in S. After localizing at a height one prime of R if necessary, we can assume that $\Phi_S \in \mathrm{Hom}_S(S^{1/p^e}, S)$ generates the Hom-set as an S^{1/p^e}-module. We may then write $\phi(\_\_) = \Phi_S(y^{1/p^e}\_\_)$ for some $y \in K(S)$. It suffices to show that $y \in S$, or in other words that $\mathrm{div}(y) \geq 0$. We still have the following diagram

$$\begin{array}{ccc} S^{1/p^e} & \xrightarrow{\phi_S} & K(S) \\ {\scriptstyle \mathrm{Tr}^{1/p^e}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ R^{1/p^e} & \xrightarrow[\phi]{} & K(R). \end{array}$$

An argument similar to the one above proves that $\mathrm{div}(y) = \pi^* D_\phi - (1-p^e)\mathrm{Ram}$ (note this requires a slight modification of the proof of Lemma 1.4). $\square$

Corollary 1.6. *Suppose that* $R \subseteq S$ *is a finite generically separable inclusion of* F*-finite normal domains with* $\pi : \mathrm{Spec}\, S \longrightarrow \mathrm{Spec}\, R$ *the induced map. Suppose* R *is* F*-split and that* $\phi : F^e_* R \longrightarrow R$ *is a Frobenius splitting such that* $\pi^*\Delta_\phi \geq \mathrm{Ram}$. *Then* S *is* F*-split as well.*

Definition 1.7. A finite inclusion of normal domains $R \subseteq S$ is *finite étale in codimension 1* if the ramification divisor $\mathrm{Ram} = 0$.

Corollary 1.8. *Suppose that* $R \subseteq S$ *is an inclusion of normal domains that is finite étale in codimension 1. If* R *is* F*-split, then so is* S.

Remark 1.9. The converse to these results does *not* hold in general, but it does hold with the additional hypothesis that $R \subseteq S$ splits (as we have seen). Note that in the case that $R \subseteq S$ is étale in codimension 1, this is equivalent to the hypothesis that $\mathrm{Tr}(S) = R$ since in that case $\mathrm{Tr} \in \mathrm{Hom}_R(S, R)$ generates the Hom-set.

There is a version that we can easily state for F-regularity as well.

Corollary 1.10. *Suppose that* $R \subseteq S$ *is a finite inclusion of normal* F*-finite domains. Fix a* $0 \neq c \in R$ *such that* R_c *and* S_c *are strongly* F*-regular. If there exists a map* $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ *such that* $\phi(F^e_* c) = 1$ *and such that* $\pi^*\Delta_\phi \geq \mathrm{Ram}$, *then* S *is strongly* F*-regular as well. In particular, if* $R \subseteq S$ *is étale in codimension 1 and* R *is strongly* F*-regular, then so is* S.

Proof. We saw earlier that if the map $R \longrightarrow F^e_* R$ which sends 1 to $F^e_* c$ splits and $R[c^{-1}]$ is strongly F-regular, then R is strongly F-regular. $\square$

References