NOTES ON CHARACTERISTIC p COMMUTATIVE ALGEBRA APRIL 7TH, 2017

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Consider the problem of measuring how singular an F-finite local ring $(R, \mathfrak{m})$ is. Based on Kunz's theorem, we should measure:

How close to free is $F^e_* R$ as an R-module.

Hilbert-Kunz multiplicity and F-signature are both attempts at quantifying that notion, asymptotically as $e \longrightarrow \infty$.

- **Hilbert-Kunz multiplicity:** Measures how many generators $F^e_* R$ has relative to the expected number if R was regular.
- **F-signature:** Measures how many free summands $F^e_* R$ has relative to the expected number if R was regular.

1. Hilbert-Kunz multiplicity

Suppose that $(R, \mathfrak{m}, k)$ is a local Noetherian ring and that M is an R-module. We let $\mu_R(M)$ denote the minimal number of generators of M as an R-module. Note that $\mu_R(M) = \ell_R(M/\mathfrak{m} \cdot M) = \mathrm{rank}_k(M/\mathfrak{m} \cdot M)$ by Nakayama's lemma.

Suppose that $R = k[\![x_1, \ldots, x_d]\!]$ and that $k = k^p$ is perfect of characteristic $p > 0$. In this case, $F^e_* R$ is a free R-module with p^{ed} generators. Because of this we make the following definition:

Definition 1.1 (Hilbert-Kunz multiplicity, perfect residue field case)**.** Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian local ring of characteristic $p > 0$ and dimension d. Suppose further that $k = k^p$ is perfect. Then we define the Hilbert-Kunz multiplicity of R to be the

$$\lim_{e \longrightarrow \infty} \frac{\ell(R/\mu^{[p^e]})}{p^{ed}} = \lim_{e \longrightarrow \infty} \frac{\mu_R(F^e_* R)}{p^{ed}}$$

if it exists. It is denoted by $e_{\mathrm{HK}}(R)$.

Example 1.2. If $R = k[\![x_1, \ldots, x_d]\!]$ and $k = k^p$ is perfect, then $e_{\mathrm{HK}}(R) = 1$.

We'll show that this limit always exists later, after we generalize this definition a bit. For now, suppose that R is the localization (at some maximal ideal) of some finite type algebra over a perfect field. If $\dim R = d$, it follows that $[F^e_* K(R) : K(R)] = p^{ed}$, and so the generic rank of $F^e_* R$ over R is p^{ed}. Hence $\mu_R(F^e_* R) \geq p^{ed}$. On the other hand if we ever had that $\mu_R(F^e_* R) = p^{ed}$, then $F^e_* R$ would be a free R-module and hence R would be regular (and then an argument similar to the one above would show that $e_{\mathrm{HK}}(R) = 1$).

Next let's figure out what to do when k is not perfect, we'll use the case of an F-finite residue field as our starting point.

Lemma 1.3. *Suppose that $(R, \mathfrak{m}, k)$ is a local ring with F-finite residue field (i.e. such that $[k : k^p] < \infty$). If M is an R-module of finite length then*

$$\ell_R(F^e_* M) = [k : k^{p^e}] \cdot \ell_{F^e_* R}(F^e_* M) = [k : k^{p^e}] \cdot \ell_R(M). \tag{1.3.1}$$

In particular, $\mu_R(F^e_* R)$, the number of generators of $F^e_* R$ as an R-module satisfies

$$\mu_R(F^e_* R) = \ell_R((F^e_* R)/\mathfrak{m}) = \ell_R\big(F^e_*(R/\mathfrak{m}^{[p^e]})\big) = [k : k^{p^e}] \cdot \ell_R\big(R/\mathfrak{m}^{[p^e]}\big) \tag{1.3.2}$$

Proof. In (1.3.1), the second equality is trivial. The first equality follows from the fact that $[k : k^{p^e}] = \ell_R(F^e_* k)$.

In (1.3.2), the first equality is just Nakayama's lemma and the second is the fact that $(R/\mathfrak{m}) \cdot F^e_* R \cong F_*(R/\mathfrak{m}^{[p^e]})$. The third equality is simply (1.3.1) applied to the finite length module $M = R/\mathfrak{m}^{[p]}$. □

On the other hand if $R = k$ is an imperfect but F-finite field, we still might want $e_{\mathrm{HK}}(R) = 1$ (since R is regular). Now, if $\mu_k(F_* k) = [F_* k : k] = [k : k^p] = n$, then $\mu_k(F^2_* k) = [F^2_* k : k] = n^2$ and more generally, $\mu_k(F^e_* k) = n^e$. Thus it is natural to try to normalize at the very least for the residue field. In particular, it would be natural to simply define

$$e_{\mathrm{HK}}(R) = \lim_{e \longrightarrow \infty} \frac{\mu_R(F^e_* R)}{[F^e_* k : k] p^{ed}} = \lim_{e \longrightarrow \infty} \frac{\ell_R(()F^e_*(R/\mathfrak{m}^{[p^e]})}{[F_* k : k]^e p^{ed}}.$$

However, based on our above lemma, this is already the same as:

$$e_{\mathrm{HK}}(R) = \lim_{e \longrightarrow \infty} \frac{\ell_R\big(R/\mathfrak{m}^{[p^e]}\big)}{p^{ed}}.$$

We take this to be our definition of Hilbert-Kunz multiplicity independent of whether or not k is perfect (*even* if k is not F-finite). At this point, there is one more generalization we will make. Instead of modding out by $\mathfrak{m}^{[p^e]}$, we fix J to be an $\mathfrak{m}$-primary ideal (ie, $\sqrt{J} = \mathfrak{m}$) and mod out by $J^{[p^e]}$.

Definition 1.4 (Hilbert-Kunz multiplicity, general case)**.** Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring of characteristic $p > 0$ and dimension d. Suppose further that J is an $\mathfrak{m}$-primary ideal. Then we define the *Hilbert-Kunz multiplicity of R along J to be*

$$e_{\mathrm{HK}}(J; R) = \lim_{e \longrightarrow \infty} \frac{\ell_R\big(R/J^{[p^e]}\big)}{p^{ed}},$$

if it exists.

Before showing it exists, let's figure out what it is for regular rings in general.

Proposition 1.5. *Suppose $(R, \mathfrak{m}, k)$ is a regular local Noetherian ring of characteristic $p > 0$ and dimension d. Then $e_{\mathrm{HK}}(J; R) = \ell(R/J)$ and in particular, $e_{\mathrm{HK}}(\mathfrak{m}; R) = e_{\mathrm{HK}}(R) = 1$.*

Proof. We first handle the case when $J = \mathfrak{m}$. Consider $\widehat{R} \cong k[\![x_1, \dots, x_d]\!]$. By construction, $\widehat{R}/(J\widehat{R})^{[p^e]} \cong R/J^{[p^e]}$, and so $e_{\mathrm{HK}}(J; R) = e_{\mathrm{HK}}(J\widehat{R}; \widehat{R})$. Thus we may assume that $R = k[\![x_1, \dots, x_d]\!]$. But clearly then $\ell_R\big(R/\mathfrak{m}^{[p^e]}\big) = p^{ed}$.

For the general case, we will show that

$$\ell_R\big(R/J^{[p^e]}\big) = p^{ed} \ell_R(R/J)$$

which will complete the proof. Consider a decomposition $0 = N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_s = R/J$ where $s = \ell_R(R/J)$ and $N_{i+1}/N_i \cong k = R/\mathfrak{m}$. Tensoring with the flat module $F^e_* R$ we obtain

$$0 = (F^e_* R) \otimes_R N_0 \subsetneq (F^e_* R) \otimes_R N_1 \subsetneq (F^e_* R) \otimes_R N_2 \subsetneq \cdots \subsetneq (F^e_* R) \otimes_R N_s \cong F^e_*(R/J^{[p^e]}).$$

Each $(F^e_* R) \otimes_R N_{i+1}/(F^e_* R) \otimes_R N_i$ is isomorphic to $F^e_*(R/\mathfrak{m}^{[p^e]})$ and so has length p^{ed} as an $F^e_* R$-module. It follows that

$$\ell_R\big(R/J^{[p^e]}\big) = p^{ed}\ell_R(R/J)$$

as desired. □

Theorem 1.6. *For $(R, \mathfrak{m})$ a Noetherian local d-dimensional ring of characteristic $p > 0$, the following are equivalent:*

(a) *R is regular.*
(b) *$\ell_R\big(R/\mathfrak{m}^{[p^e]}\big) = p^{ed}$ for every $e > 0$.*
(c) *$\ell_R\big(R/\mathfrak{m}^{[p^e]}\big) = p^{ed}$ for some $e > 0$.*

Proof. We just showed that (a) $\Leftarrow$ (b) and obviously (b) $\Leftarrow$ (c), so it suffices to show that (c) implies (a). We essentially already sketched this when the residue field is perfect (since then (c) implies that $F^{ed}_* R$ is free). The general case is left as an exercise (if time permits, we may prove a more general theorem later showing that $e_{\mathrm{HK}}(R) = 1$ actually implies that R is regular). □

Exercise 1.1. Prove Theorem 1.6.

Before moving on to existence, let me make one more observation.

Proposition 1.7. *With notation as above*

$$p^{ad} e_{\mathrm{HK}}(J; R) = e_{\mathrm{HK}}(J^{[p^a]}; R).$$

Proof. It is obvious. □

References