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Reconciling Riemann-Roch results

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In honor of the contributions of Wolmer Vasconcelos

Abstract. In the course of their work on the homological conjectures, Peskine and Szpiro proved a Riemann-Roch formula for graded modules; we show that this agrees with the Hirzebruch-Riemann-Roch formula in the case of graded modules over polynomial rings.

In 1974 Peskine and Szpiro [**PS**] proved a number of conjectures on intersection multiplicities of graded modules, using a formula that they developed for the Hilbert polynomials of such modules. This formula was considered to be a kind of Riemann-Roch formula, and indeed it was one of the inspirations for the local Chern characters and the Riemann-Roch formula for local rings, developed by Baum, Fulton, and MacPherson [**BFM**].

In this paper we do not discuss the connections to questions on multiplicities, but look instead at the formula of Peskine and Szpiro and examine the extent to which it may be considered a Riemann-Roch formula; in other words, we compare it to the Riemann-Roch formula of Hirzebruch. In the case of perfect complexes over polynomial rings, we show that the formulae agree in a precise sense.

1. The Peskine-Szpiro formula

Let A be an $\mathbb{N}$-graded ring such that A_0 is a field $\mathbb{K}$, and A is generated over A_0 by finitely many elements of A_1. For n an integer, $A(n)$ will denote the module A with the shifted grading $A(n)_k = A_{n+k}$ for each k. By a *perfect complex* $F_\bullet$ we mean a bounded complex

$$0 \longrightarrow F_s \longrightarrow F_{s-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0\,, \tag{1.1}$$

where each F_i is a finite direct sum of copies of $A(n)$—for varying n—and such that the homomorphisms in $F_\bullet$ preserve degrees. Set

$$F_i \;=\; \bigoplus_{j=1}^{\beta_i} A(n_{ij}) \qquad \text{for each } i\,.$$

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Recall that if M is a finitely generated graded A-module, the Hilbert polynomial of M is the polynomial $P_M(x)$ with the property that $P_M(n)$ agrees with $\operatorname{rank}_{\mathbb{K}} M_n$ for sufficiently large integers n. The Peskine-Szpiro result is a formula for the alternating sum of the Hilbert polynomials of the homology modules of $F_\bullet$ in terms of the integers n_{ij}:

For each integer $k \geqslant 0$, set

$$\rho_k \;=\; \frac{1}{k!}\sum_{i=0}^{s}(-1)^i\sum_{j=1}^{\beta_i} n_{ij}^k\,.$$

Then the Peskine-Szpiro formula is

$$\sum_{i=0}^{s}(-1)^i P_{H_i}(x) \;=\; \sum_{k\geqslant 0}\rho_k P_A^{(k)}(x)\,, \tag{1.2}$$

where H_i denotes the i-th homology module of the complex $F_\bullet$ and $P_A^{(k)}(x)$ is the k-th derivative of the polynomial $P_A(x)$.

2. Chern classes

We briefly review some material that may be found in [**Ha**] or [**Fu**]. Let X be a nonsingular projective variety of dimension d over a field $\mathbb{K}$. A *cycle* of codimension k on X is an element of the free abelian group generated by closed irreducible subvarieties of X having codimension k. The group $\mathrm{CH}^k(X)$ consists of cycles of codimension k modulo rational equivalence. Cycles of codimension d have the form $\sum_i n_iP_i$ for points P_i of X, and one has a group homomorphism

$$\deg\colon\ \mathrm{CH}^d(X)\longrightarrow\mathbb{Z}\qquad\text{where }\ \deg\sum_i n_iP_i=\sum_i n_i\,. \tag{2.1}$$

The intersection pairing on X provides

$$\bigoplus_{r=0}^{d}\mathrm{CH}^r(X)$$

with the structure of a commutative ring, the *Chow ring* of X, denoted $\mathrm{CH}(X)$. Extending the correspondence between invertible sheaves and divisors, for each locally free sheaf $\mathcal{F}$ on X—say of rank r—there exist *Chern classes* $c_i = c_i(\mathcal{F})$ in $\mathrm{CH}^i(X)$, where $c_0 = 1$ and $c_i = 0$ for all $i > r$. The *Chern polynomial* of $\mathcal{F}$ is

$$c_t(\mathcal{F}) \;=\; 1 + c_1t + c_2t^2 + \cdots + c_rt^r\,.$$

For an exact sequence of locally free sheaves

$$0\longrightarrow\mathcal{F}'\longrightarrow\mathcal{F}\longrightarrow\mathcal{F}''\longrightarrow 0\,, \tag{2.2}$$

the *Whitney sum formula* states that

$$c_t(\mathcal{F}) \;=\; c_t(\mathcal{F}')c_t(\mathcal{F}'')\,.$$

For the purposes of a Riemann-Roch formula, one wants to associate to locally free sheaves, invariants that are additive on exact sequences. Towards this, factor the polynomial $c_t(\mathcal{F})$ formally as

$$c_t(\mathcal{F}) \;=\; \prod_{i=1}^{r}(1+\alpha_it)\,;$$

the α_i are the *Chern roots* of $\mathcal{F}$. Working in $\mathrm{CH}(X)_{\mathbb{Q}} = \mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, define the *Chern character* of $\mathcal{F}$ as

$$\mathrm{ch}(\mathcal{F}) \;=\; \sum_{i=1}^{r} e^{\alpha_i}\,, \qquad \text{where } e^x \;=\; 1 + x + \frac{1}{2}x^2 + \cdots\,.$$

Since $\mathrm{ch}(\mathcal{F})$ is a symmetric function of the Chern roots $\alpha_1, \ldots, \alpha_r$, and the Chern classes $c_1, \ldots, c_r$ are precisely the elementary symmetric polynomials in $\alpha_1, \ldots, \alpha_r$, one can express $\mathrm{ch}(\mathcal{F})$ in terms of the Chern classes; the first few terms, as may be found in [**Ha**, page 432] or [**Fu**, Example 3.2.3], are

$$\mathrm{ch}(\mathcal{F}) \;=\; r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots\,.$$

The Whitney sum formula now yields

$$\mathrm{ch}(\mathcal{F}) \;=\; \mathrm{ch}(\mathcal{F}') + \mathrm{ch}(\mathcal{F}'')\,,$$

i.e., Chern characters are additive on short exact sequences.

The Chern character of a tensor product of locally free sheaves is

$$\mathrm{ch}(\mathcal{F} \otimes \mathcal{F}') \;=\; \mathrm{ch}(\mathcal{F})\,\mathrm{ch}(\mathcal{F}')\,. \tag{2.3}$$

The formal power series

$$\frac{1 - e^{-x}}{x} \;=\; 1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \cdots$$

is an invertible element of $\mathbb{Q}[[x]]$; we denote its inverse by

$$Q(x) \;=\; x/(1 - e^{-x})\,. \tag{2.4}$$

The occurrence of this power series will be justified in Example 3.1. For now, we conclude this section with one last definition: the *Todd class* of a locally free sheaf $\mathcal{F}$ with Chern roots $\alpha_1, \ldots, \alpha_r$ is

$$\mathrm{td}(\mathcal{F}) \;=\; \prod_{i=1}^{r} Q(\alpha_i)\,.$$

Once again, this is a symmetric function of $\alpha_1, \ldots, \alpha_r$, so it can be expressed in terms of the Chern classes; indeed, one has

$$\mathrm{td}(E) \;=\; 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}(c_1c_2) + \cdots\,.$$

For an exact sequence (2.2), the Whitney sum formula implies

$$\mathrm{td}(\mathcal{F}) \;=\; \mathrm{td}(\mathcal{F}')\,\mathrm{td}(\mathcal{F}'')\,. \tag{2.5}$$

3. The Riemann-Roch Theorem

Let $\mathcal{F}$ be a locally free sheaf on a nonsingular projective variety X of dimension d. The *Euler characteristic* of $\mathcal{F}$ is the alternating sum of the ranks of the sheaf cohomology groups $H^i(X, \mathcal{F})$, i.e.,

$$\chi(\mathcal{F}) \;=\; \sum_{i=0}^{d} (-1)^i \operatorname{rank}_{\mathbb{K}} H^i(X, \mathcal{F})\,.$$

Let $\mathcal{T}_X$ be the tangent sheaf of X. The Riemann-Roch theorem of Hirzebruch states

$$\chi(\mathcal{F}) \;=\; \deg\,[\mathrm{ch}(\mathcal{F})\,\mathrm{td}(\mathcal{T}_X)]_d\,,$$

where $[-]_d$ denotes the component in $\mathrm{CH}^d(X)_{\mathbb{Q}}$ and $\deg\colon \mathrm{CH}^d(X)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ extends the homomorphism $\deg\colon \mathrm{CH}^d(X) \longrightarrow \mathbb{Z}$ of (2.1); see [**Ha**, Appendix A].

Let $\mathcal{L}_\bullet$ be a bounded complex of locally free sheaves on X, say

$$0 \longrightarrow \mathcal{L}_s \longrightarrow \mathcal{L}_{s-1} \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow 0\,.$$

The Euler characteristic of $\mathcal{L}_\bullet$ is

$$\chi(\mathcal{L}_\bullet) \;=\; \sum_{i=0}^{s}(-1)^i\chi(\mathcal{L}_i)\,.$$

Setting $\mathrm{ch}(\mathcal{L}_\bullet) = \sum_i (-1)^i \,\mathrm{ch}(\mathcal{L}_i)$, the Riemann-Roch theorem takes the form

$$\chi(\mathcal{L}_\bullet) \;=\; \deg\,[\mathrm{ch}(\mathcal{L}_\bullet)\,\mathrm{td}(\mathcal{T})]_d\,.$$

Example 3.1. We examine the Riemann-Roch theorem in the case X is projective space $\mathbb{P}^d$, and provide some justification for the choice of the power series $Q(x)$ used in the definition of the Todd class; see (2.4).

Any subvariety of degree k in $\mathbb{P}^d$ is linearly equivalent to k times a linear space of the same dimension, so

$$\mathrm{CH}(\mathbb{P}^d)_{\mathbb{Q}} \;=\; \mathbb{Q}[h]/(h^{d+1})\,,$$

where h is the class of a hyperplane; the class of a point in $\mathbb{P}^d$ is identified with

$$h^d \;\in\; \mathrm{CH}^d(\mathbb{P}^d)_{\mathbb{Q}}\,.$$

By the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^d} \longrightarrow \mathcal{O}_{\mathbb{P}^d}(1)^{d+1} \longrightarrow \mathcal{T}_{\mathbb{P}^d} \longrightarrow 0$$

and (2.5), one has

$$\mathrm{td}(\mathcal{T}_{\mathbb{P}^d}) \;=\; \mathrm{td}\left(\mathcal{O}_{\mathbb{P}^d}(1)^{d+1}\right) \;=\; \mathrm{td}(h)^{d+1}\,.$$

Taking $\mathcal{F}$ to be $\mathcal{O}_{\mathbb{P}^d}$ in the Riemann-Roch theorem, we see that

$$1 \;=\; \deg\,[\mathrm{td}(\mathcal{T}_{\mathbb{P}^d})]_d \;=\; \deg\,[\mathrm{td}(h)^{d+1}]_d\,.$$

Thus, $\mathrm{td}(h)$ is a power series such that x^d occurs with unit coefficient in $\mathrm{td}(h)^{d+1}$. It turns out that

$$Q(x) \;=\; x/(1-e^{-x})$$

is the only power series in $\mathbb{Q}[[x]]$ with the property that x^d occurs with unit coefficient in $Q(x)^{d+1}$ for each $d \geqslant 0$; this can be proved using the recursion (4.3).

It follows from the above discussion that

$$\mathrm{td}(\mathcal{T}_{\mathbb{P}^d}) \;=\; Q(h)^{d+1}\,.$$

4. A comparison

We now reconcile the Riemann-Roch theorem of Hirzebruch with the formula of Peskine and Szpiro; first, some notation:

We use $\underset{k}{\complement}\, p(x)$ to denote the coefficient of x^k in a polynomial or formal power series $p(x)$.

Let A be a standard graded polynomial ring over a field. A perfect complex of A-modules (1.1) defines a complex of locally free sheaves on $\operatorname{Proj} A$, which is $\mathbb{P}^d$. The sheaves that occur are direct sums of $\mathcal{O}_{\mathbb{P}^d}(n)$ for varying n.

We now translate the Peskine-Szpiro formula into these terms. The modules $F_i = \bigoplus_j A(n_{ij})$ in (1.1) define sheaves $\bigoplus_j \mathcal{O}_{\mathbb{P}^d}(n_{ij})$ on $\mathbb{P}^d$. The complex $F_\bullet$ thus yields a complex $\mathcal{L}_\bullet$ of locally free sheaves $\mathcal{L}_i$, where

$$\mathcal{L}_i \;=\; \bigoplus_{j=1}^{\beta_i} \mathcal{O}_{\mathbb{P}^d}(n_{ij})\,.$$

The Chern polynomial of $\mathcal{O}_{\mathbb{P}^d}(1)$ is $1+ht$, for h the class of a hyperplane. Hence

$$\mathrm{ch}(\mathcal{O}_{\mathbb{P}^d}(1)) \;=\; e^h\,,$$

and using (2.3), it follows that the Chern character of $\mathcal{O}_{\mathbb{P}^d}(n)$ is

$$\begin{aligned}\mathrm{ch}(\mathcal{O}_{\mathbb{P}^d}(n)) \;&=\; e^{nh}\\ &=\; 1+nh+\frac{n^2h^2}{2!}+\frac{n^3h^3}{3!}+\cdots+\frac{n^dh^d}{d!} \qquad \text{in } \mathbb{Q}[h]/(h^{d+1})\,.\end{aligned}$$

Since Chern characters are additive on short exact sequences—in particular, on direct sums—the Chern character of $\mathcal{L}_\bullet$ is

$$\begin{aligned}\mathrm{ch}(\mathcal{L}_\bullet) \;&=\; \sum_{i=0}^{s}(-1)^i\,\mathrm{ch}(\mathcal{L}_i)\\ &=\; \sum_{i=0}^{s}(-1)^i\sum_{j=1}^{\beta_i}\mathrm{ch}(\mathcal{O}_X(n_{ij}))\\ &=\; \sum_{i=0}^{s}(-1)^i\sum_{j=1}^{\beta_i}\big(1+n_{ij}h+\frac{n_{ij}^2h^2}{2!}+\cdots+\frac{n_{ij}^dh^d}{d!}\big)\,.\end{aligned}$$

Collecting coefficients, we obtain that for each k the coefficient of h^k is

$$\sum_{i=0}^{s}(-1)^i\sum_{j=1}^{\beta_i}\frac{n_{ij}^k}{k!}\,.$$

Since this is precisely ρ_k, we see that

$$\mathrm{ch}(\mathcal{L}_\bullet) \;=\; \rho_0+\rho_1h+\rho_2h^2+\cdots+\rho_dh^d\,.$$

Thus, the quantities ρ_k in the Peskine-Szpiro formula occur as the components of the Chern character in the Riemann-Roch Theorem. We now look at the other half of the theorem, namely the Todd class.

In the Riemann-Roch Theorem for projective space $\mathbb{P}^d$, the Todd class of the tangent bundle is $\mathrm{td}(\mathcal{T}_{\mathbb{P}^d}) = Q(h)^{d+1}$, where $Q(x) = x/(1-e^{-x})$; see Example 3.1. Since $h^{d+1}=0$, this may be written as

$$\mathrm{td}(\mathcal{T}_{\mathbb{P}^d}) \;=\; a_0+a_1h+\cdots+a_dh^d\,,$$

where the a_k are rational numbers. As $\deg h^d = 1$ in $\mathrm{CH}^d(\mathbb{P}^d)_{\mathbb{Q}}$, the Riemann-Roch Theorem states that

$$\chi(\mathcal{L}_\bullet) \;=\; \rho_da_0+\rho_{d-1}a_1+\cdots+\rho_0a_d\,,\ \text{where } a_k = \underset{k}{\complement}\,Q(x)^{d+1}\,.$$

To compare this with the Peskine-Szpiro formula (1.2), we need to know the relation between the Hilbert polynomial of $A(n)$ and the Euler characteristic of

$\mathcal{O}_{\mathbb{P}^d}(n)$. This is very simple: if M is a finitely generated graded A-module with Hilbert polynomial $P_M(x)$ and $\mathcal{F} = \widetilde{M}$ the associated coherent sheaf, then

$$P_M(n) \;=\; \chi(\mathcal{F}(n)) \qquad \text{for all integers } n\,,$$

where $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^d}(n)$. In particular, the Euler characteristic $\chi(\mathcal{F})$ equals $P_M(0)$. Hence, the Peskine-Szpiro formula yields

$$\chi(\mathcal{L}_\bullet) \;=\; \sum_{i=0}^{s}(-1)^i P_{H_i}(0) \;=\; \sum_{k\geqslant 0}\rho_k P_A^{(k)}(0)\,.$$

Setting $b_i = P_A^{(d-i)}(0)$ for $i = 0,\dots,d$, we have

$$\chi(\mathcal{L}_\bullet) \;=\; \rho_d b_0 + \rho_{d-1}b_1 + \cdots + \rho_0 b_d\,.$$

In this formula, b_k is the constant term of the $(d-k)$-th derivative of the polynomial $P_A(x)$, i.e.,

$$b_k \;=\; (d-k)!\,\underset{d-k}{\mathsf{C}}\,P_A(x)\,.$$

Since A is the polynomial ring in $d+1$ variables, its Hilbert polynomial is $\binom{x+d}{d}$, and so

$$b_k \;=\; (d-k)!\,\underset{d-k}{\mathsf{C}}\binom{x+d}{d}\,.$$

Thus, we have two similar formulae for the Euler characteristic of $\mathcal{L}_\bullet$, involving the sequences $a_0, a_1, \dots, a_d$ and $b_0, b_1, \dots, b_d$; the first sequence is given by the first $d+1$ coefficients of the power series $Q(x)^{d+1}$, where $Q(x) = x/(1-e^{-x})$; the second sequence is derived from the coefficients of the polynomial $\binom{x+d}{d}$. It is by no means *a priori* obvious that these sequences are related; the remainder of this section is devoted to giving a direct proof that these sequences are indeed the same.

Proposition 4.1. *Let d be a nonnegative integer. Then, for each integer k with $0 \leqslant k \leqslant d$, the coefficient of x^k in the formal power series*

$$\left(\frac{x}{1-e^{-x}}\right)^{d+1}$$

agrees with the coefficient of x^{d-k} in the polynomial

$$(d-k)!\binom{x+d}{d} \;=\; \frac{(d-k)!}{d!}(x+1)(x+2)\cdots(x+d)\,.$$

Proof. Set

$$q(d,k) \;=\; (d-k)!\,\underset{d-k}{\mathsf{C}}\binom{x+d}{d}\,.$$

Note that $q(0,0) = 1$, and that $q(d,k) = 0$ unless $0 \leqslant k \leqslant d$. We claim that

$$q(d,k) \;=\; q(d-1,k-1) + \frac{d-k}{d}q(d-1,k)\,. \tag{4.1}$$

This holds since

$$\begin{aligned}
&q(d-1,k-1) \;+\; \frac{d-k}{d}q(d-1,k)\\
&\quad=\; (d-k)!\,\mathsf{C}_{d-k}\binom{x+d-1}{d-1} \;+\; \frac{d-k}{d}(d-k-1)!\,\mathsf{C}_{d-k-1}\binom{x+d-1}{d-1}\\
&\quad=\; \frac{(d-k)!}{(d-1)!}\mathsf{C}_{d-k}(x+1)\cdots(x+d-1) \;+\; \frac{(d-k)!}{d!}\mathsf{C}_{d-k-1}(x+1)\cdots(x+d-1)\\
&\quad=\; \frac{(d-k)!}{d!}\mathsf{C}_{d-k}(x+1)\cdots(x+d-1)d \;+\; \frac{(d-k)!}{d!}\mathsf{C}_{d-k}(x+1)\cdots(x+d-1)x\\
&\quad=\; \frac{(d-k)!}{d!}\mathsf{C}_{d-k}(x+1)\cdots(x+d-1)(x+d)\\
&\quad=\; q(d,k)\,.
\end{aligned}$$

Next, consider the polynomials

$$q_d \;=\; \sum_k q(d,k)x^k\,, \qquad \text{where } d \geqslant 0\,.$$

Using the recursion relation (4.1), it follows that

$$\begin{aligned}
q_d \;&=\; \sum_k q(d-1,k-1)x^k \;+\; \frac{d-k}{d}\sum_k q(d-1,k)x^k\\
&=\; x\sum_k q(d-1,k-1)x^{k-1} \;+\; \sum_k q(d-1,k)x^k \;-\; \frac{kx}{d}\sum_k q(d-1,k)x^{k-1}\\
&=\; xq_{d-1} \;+\; q_{d-1} \;-\; \frac{x}{d}q'_{d-1}\,,
\end{aligned}$$

where q'_{d-1} denotes the derivative of q_{d-1} with respect to x. Thus, we have

$$q_d \;=\; (1+x)q_{d-1} - \frac{x}{d}q'_{d-1} \quad \text{for } d \geqslant 1\,, \qquad \text{and } q_0 = 1\,. \tag{4.2}$$

The polynomials q_d—and hence the numbers $q(d,k)$—are determined by (4.2).

Next, we claim that the formal power series

$$Q_d \;=\; \left(\frac{x}{1-e^{-x}}\right)^{d+1}\,, \qquad \text{where } d \geqslant 0\,,$$

satisfy a similar recursion, namely

$$Q_d \;=\; (1+x)Q_{d-1} - \frac{x}{d}Q'_{d-1} \quad \text{for } d \geqslant 1\,. \tag{4.3}$$

The derivative of $x/(1-e^{-x})$ may be computed as

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x}{1-e^{-x}}\right) \;&=\; \frac{d}{dx}\left(\frac{1-e^{-x}}{x}\right)^{-1}\\
&=\; -\left(\frac{1-e^{-x}}{x}\right)^{-2}\left(\frac{xe^{-x}-1+e^{-x}}{x^2}\right)\\
&=\; \frac{1-e^{-x}-xe^{-x}}{(1-e^{-x})^2}\,.
\end{aligned}$$

Hence

$$\begin{aligned}(1+x)Q_{d-1} - \frac{x}{d}Q'_{d-1} &= (1+x)\left(\frac{x}{1-e^{-x}}\right)^d - \frac{x}{d}\frac{d}{dx}\left(\frac{x}{1-e^{-x}}\right)^d \\ &= (1+x)\left(\frac{x}{1-e^{-x}}\right)^d - x\left(\frac{x}{1-e^{-x}}\right)^{d-1}\left(\frac{1-e^{-x}-xe^{-x}}{(1-e^{-x})^2}\right) \\ &= \left(\frac{x}{1-e^{-x}}\right)^d\left(1+x - \frac{1-e^{-x}-xe^{-x}}{1-e^{-x}}\right) \\ &= \left(\frac{x}{1-e^{-x}}\right)^{d+1},\end{aligned}$$

which proves (4.3).

The proposition asserts that the coefficients of x^k in Q_d and q_d agree for each k with $0 \leqslant k \leqslant d$, i.e., that

$$Q_d - q_d \in \left(x^{d+1}\right)\mathbb{Q}[[x]].$$

We prove this by induction on d; the case $d = 0$ is readily checked. Assuming the result for $d-1$, we have

$$Q_{d-1} - q_{d-1} = x^d E \qquad \text{for some } E \in \mathbb{Q}[[x]].$$

But then, using (4.3) and (4.2), we have

$$\begin{aligned}Q_d - q_d &= (1+x)\big(Q_{d-1} - q_{d-1}\big) - \frac{x}{d}\big(Q'_{d-1} - q'_{d-1}\big) \\ &= (1+x)x^d E - \frac{x}{d}\big(x^d E\big)' \\ &= (1+x)x^d E - \frac{x}{d}\big(dx^{d-1}E + x^d E'\big) \\ &= x^{d+1}\big(E - \frac{1}{d}E'\big).\end{aligned}$$

□

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