# POSITIVE CHARACTERISTIC METHODS IN COMMUTATIVE ALGEBRA

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These are notes of ten lectures given at IIT Bombay in June 2017 on applications of *tight closure theory*. The theory was developed by Melvin Hochster and Craig Huneke in the paper [HH2]; excellent accounts may be found in Huneke's CBMS lecture notes [Hu], and Hochster's course notes [Ho5].

At this stage, it has become a formidable task to give anything close to a comprehensive treatment of tight closure theory; our intention here is only to talk about a few concrete applications, and emphasize the concise and elegant proofs that the theory offers.

These notes focus on applications to the Anand-Dumir-Gupta Conjectures about magic squares, the Hochster-Roberts Theorem regarding the Cohen-Macaulay property of rings of invariants, the Briançon-Skoda Theorem in complex analytic geometry, and uniform bounds on symbolic powers of ideals due to Ein-Lazarsfeld-Smith and Hochster-Huneke.

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## 1. Magic squares

A *magic square* is a matrix with nonnegative integer entries such that each row and each column has the same sum, called the *line sum*. Let $H_n(r)$ denote the number of $n \times n$ magic squares with line sum $r$. Then

$$H_n(0) = 1, \qquad H_n(1) = n! \qquad \text{and} \qquad \sum_{n \geqslant 0} \frac{H_n(2)x^n}{(n!)^2} = \frac{e^{x/2}}{\sqrt{1-x}},$$

where the first two formulae are elementary, and the third was proved by Anand, Dumir, and Gupta [ADG]. On the other hand, viewing $H_n(r)$ as a function of $r \geqslant 0$, one has

$$H_1(r) = 1, \qquad H_2(r) = r+1, \qquad \text{and} \qquad H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$

Once again, the first two are trivial, bearing in mind that $H_2(r)$ counts the matrices

$$\begin{pmatrix} i & r-i \\ r-i & i \end{pmatrix}, \qquad 0 \leqslant i \leqslant r,$$

while the formula for $H_3(r)$ is due to MacMahon [Mac, § 407]; we compute it here in Example 1.2. It was conjectured[1] in [ADG] that $H_n(r)$ agrees with a degree $(n-1)^2$ polynomial in $r$ for all integers $r \geqslant 0$. This—and more—was proved by Stanley [St1]; see also [St2, St3]. We start with a tight closure proof of the following:

**Theorem 1.1.** *Let $H_n(r)$ be the number of $n \times n$ magic squares with line sum $r$. Then $H_n(r)$ agrees with a degree $(n-1)^2$ polynomial in $r$ for all integers $r \geqslant 0$.*

**Counting using rings.** Let $\mathbb{F}$ be a field. Let $(x_{ij})$ be an $n \times n$ matrix of indeterminates, for $n$ a fixed positive integer, and set $R$ to be the polynomial ring

$$R := \mathbb{F}[x_{ij} \mid 1 \leqslant i, j \leqslant n].$$

Set $S$ to be the $\mathbb{F}$-subalgebra of $R$ generated by the monomials

$$\prod_{i,j} x_{ij}^{a_{ij}} \quad \text{such that } \big(a_{ij}\big) \text{ is an } n \times n \text{ magic square.}$$

For example, in the case $n = 2$, the magic squares are

$$\begin{pmatrix} i & r-i \\ r-i & i \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (r-i) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad 0 \leqslant i \leqslant r,$$

and it follows that

$$\begin{aligned} S &= \mathbb{F}[x_{11}^i \, x_{12}^{r-i} \, x_{21}^{r-i} \, x_{22}^i \mid 0 \leqslant i \leqslant r] \\ &= \mathbb{F}[(x_{11}x_{22})^i \, (x_{12}x_{21})^{r-i} \mid 0 \leqslant i \leqslant r] = \mathbb{F}[x_{11}x_{22}, \, x_{12}x_{21}]. \end{aligned}$$

[1] Or, at least, "It thus appears that . . . "

More generally, the Birkhoff-von Neumann theorem yields that each magic square is a sum of permutation matrices. Thus, $S$ is generated over $\mathbb{F}$ by the $n!$ monomials

$$\prod_{i=1}^{n} x_{i\sigma(i)} \quad \text{for } \sigma \text{ a permutation of } \{1,\dots,n\}.$$

Consider the $\mathbb{Q}$-grading on $R$ with $R_0 := \mathbb{F}$ and $\deg x_{ij} := 1/n$ for each $i, j$. Then

$$\deg\Big(\prod_{i=1}^{n} x_{i\sigma(i)}\Big) \;=\; 1,$$

so $S$ is a *standard graded* $\mathbb{F}$-algebra, i.e., $S$ is $\mathbb{N}$-graded with $S_0 = \mathbb{F}$ and $S = \mathbb{F}[S_1]$. Let

$$H_S(t) := \sum_{r \geqslant 0} (\operatorname{rank}_{\mathbb{F}} S_r)\, t^r$$

which is the *Hilbert series* of $S$. Then $H_n(r)$ is the coefficient of $t^r$ in the series $H_S(t)$.

**Example 1.2.** In the case $n = 3$, the permutation matrices satisfy the linear relation

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} + \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$
$$= \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} + \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},$$

so $S$ has a presentation

$$S \;=\; \mathbb{F}[y_1,y_2,y_3,y_4,y_5,y_6]\,/\,(y_1y_2y_3 - y_4y_5y_6)$$

where

$$\begin{aligned} &y_1 \longmapsto x_{11}x_{22}x_{33}, && y_2 \longmapsto x_{12}x_{23}x_{31}, && y_3 \longmapsto x_{13}x_{21}x_{32},\\ &y_4 \longmapsto x_{13}x_{22}x_{31}, && y_5 \longmapsto x_{11}x_{23}x_{32}, && y_6 \longmapsto x_{12}x_{21}x_{33}. \end{aligned}$$

Since $S$ is a hypersurface of degree 3, its Hilbert series is

$$H_S(t) \;=\; \frac{1-t^3}{(1-t)^6} \;=\; \frac{1+t+t^2}{(1-t)^5}.$$

The coefficient of $t^r$ in this series is readily seen to be

$$H_3(r) \;=\; \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}. \qquad \blacksquare$$

**Exercise 1.3.** Determine the number of $3 \times 3$ magic squares, with line sum $r$, of the form

$$\begin{pmatrix}0&-&-\\-&-&-\\-&-&-\end{pmatrix}.$$

**Exercise 1.4.** Determine the number of $3 \times 3$ *symmetric* magic squares with line sum $r$.

**Hilbert series.** Returning to the case of $n \times n$ magic squares, since the ring $S$ generated by monomials corresponding to permutation matrices is standard graded, one has

$$H_S(t) = \frac{f(t)}{(1-t)^d} \tag{1.4.1}$$

where $d = \dim S$ and $f(t)$ is a polynomial with $f(1) \neq 0$. Since $H_n(r)$ is the coefficient of $t^r$ in $H_S(t)$, it follows that $H_n(r)$ agrees with a degree $d-1$ polynomial in $r$ for *large* positive integers $r$. It remains to compute the dimension of the ring $S$, and to show that $H_n(r)$ agrees with a polynomial in $r$ for *each* integer $r \geqslant 0$.

The dimension of $S$ may be computed as the transcendence degree of the fraction field of $S$ over $\mathbb{F}$; this equals the number of monomials in $S$ that are algebraically independent over $\mathbb{F}$. Since monomials are algebraically independent precisely if their exponent vectors are linearly independent, the dimension $d$ of $S$ is the rank of the $\mathbb{Q}$-vector space spanned by the $n \times n$ magic squares. The rank of this vector space may be computed by counting the choices "$-$" that may be made when forming an $n \times n$ matrix over $\mathbb{Q}$ with constant line sum; the remaining entries "$*$" below are then forced:

$$\begin{pmatrix} - & - & \cdots & - & - \\ - & - & \cdots & - & * \\ - & - & \cdots & - & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ - & - & \cdots & - & * \\ * & * & * & * & * \end{pmatrix}.$$

Hence $d = (n-1)^2 + 1$. It follows that $H_n(r)$ agrees with a degree $(n-1)^2$ polynomial in $r$ for *large* positive integers $r$. To see that $H_n(r)$ agrees with a polynomial in $r$ for *each* $r \geqslant 0$, it suffices to show that the degree of the polynomial $f(t)$ in (1.4.1) is at most $d-1$; indeed, if this is the case, we may write

$$f(t) \ = \ \sum_{i=0}^{d-1} a_i (1-t)^i$$

with $a_i \in \mathbb{Q}$, and so

$$H_S(t) \ = \ \sum_{i=0}^{d-1} \frac{a_i}{(1-t)^{d-i}} \ = \sum_{i=1}^{d} \frac{a_{d-i}}{(1-t)^i},$$

which is a power series where the coefficient of $t^r$ agrees with a polynomial in $r$ for each integer $r \geqslant 0$. We shall prove that $\deg f(t) \leqslant d-1$, and develop the requisite tight closure theory along the way.

Consider the $\mathbb{F}$-linear map $\rho\colon R \longrightarrow S$ that fixes a monomial $\prod x_{ij}^{a_{ij}}$ if $(a_{ij})$ is a magic square, and maps it to 0 otherwise. Since the sum of two magic squares is a magic square, and the sum of a magic square and a non-magic square is a non-magic square, $\rho$ is a homomorphism of $S$-modules. As $\rho$ fixes $S$, it is an $S$-linear splitting of the inclusion $S \subseteq R$, which implies that $S$ is a direct summand of $R$ as an $S$-module.

Thus far, the field $\mathbb{F}$ was arbitrary; for the rest of this section, assume $\mathbb{F}$ is an algebraically closed field of positive characteristic $p$. Since it suffices for the goal at hand, we will also assume that all rings, ideals, and elements in question are homogeneous.

**Definition 1.5.** Let $A$ be a domain of characteristic $p > 0$, and let $q$ denote a varying positive integer power of $p$. For an ideal $\mathfrak{a}$ of $A$, define

$$\mathfrak{a}^{[q]} := (a^q \mid a \in \mathfrak{a}).$$

The *tight closure* of $\mathfrak{a}$ is the ideal

$$\mathfrak{a}^* := \left\{ z \in A \mid \text{ there exists a nonzero } c \text{ in } A \text{ with } cz^q \in \mathfrak{a}^{[q]} \text{ for each } q = p^e \right\}.$$

While the results hold in greater generality, the following will suffice for now:

**Lemma 1.6.** *Let $R$ be a polynomial ring over an algebraically closed field $\mathbb{F}$. Let $S$ be an $\mathbb{F}$-subalgebra of $R$ that is a direct summand of $R$ as an $S$-module. Then:*

(1) *For each homogeneous ideal $\mathfrak{a}$ of $R$, one has $\mathfrak{a}^* = \mathfrak{a}$.*
(2) *For each homogeneous ideal $\mathfrak{a}$ of $S$, one has $\mathfrak{a}^* = \mathfrak{a}$.*
(3) *The ring $S$ is Cohen-Macaulay, i.e., each homogeneous system of parameters for $S$ is a regular sequence on $S$.*
(4) *If $\boldsymbol{y} = y_1, \dots, y_d$ is a homogeneous system of parameters for $S$ consisting of elements of degree 1, then $S_{\geqslant d} \subseteq \boldsymbol{y}S$.*

*Proof of Theorem 1.1.* The ring $S$ is standard graded, with $S_0$ an algebraically closed field. Hence $S$ indeed has a homogeneous system of parameters $\boldsymbol{y}$ consisting of degree 1 elements. Using (3), it follows that

$$H_S(t) \ = \ \frac{H_{S/\boldsymbol{y}S}(t)}{(1-t)^d}.$$

The polynomial $f(t) = H_{S/\boldsymbol{y}S}(t)$ has degree at most $d-1$ by (4). Thus, the lemma above completes the proof. □

*Proof of Lemma 1.6.* (1) Let $z$ be an element of $\mathfrak{a}^*$. Without loss of generality, assume $z$ is homogeneous. Then there exists a nonzero homogeneous element $c$ of positive degree such that $cz^q \in \mathfrak{a}^{[q]}$ for each $q = p^e$. Let $R = \mathbb{F}[x_1, \dots, x_m]$, in which case the ring consisting of $q$-th roots of elements of $R$, is

$$R^{1/q} := \mathbb{F}[x_1^{1/q}, \dots, x_m^{1/q}].$$

Since $R^{1/q}$ is a free $R$-module with basis

$$x_1^{i_1/q} \cdots x_m^{i_m/q}$$

where $0 \leqslant i_j \leqslant q-1$, one has

$$(\mathfrak{a}R^{1/q} :_{R^{1/q}} z) \ = \ (\mathfrak{a} :_R z)R^{1/q}.$$

Taking $q$-th roots in an equation exhibiting $cz^q \in \mathfrak{a}^{[q]}$ one sees that $c^{1/q}z \in \mathfrak{a}R^{1/q}$, so

$$c^{1/q} \in (\mathfrak{a}R^{1/q} :_{R^{1/q}} z) = (\mathfrak{a} :_R z)R^{1/q}.$$

But then the ideal $(\mathfrak{a} :_R z)R^{1/q}$ contains elements of arbitrarily small positive degree, which is only possible if $(\mathfrak{a} :_R z)$ equals $R$.

(2) If $z \in \mathfrak{a}^*$ for a homogeneous ideal $\mathfrak{a}$ of $S$, then $z \in (\mathfrak{a}R)^*$. But $(\mathfrak{a}R)^* = \mathfrak{a}R$ by (1), so $z$ belongs to $\mathfrak{a}R \cap S$. This ideal equals $\mathfrak{a}$ since $S$ is a direct summand of $R$.

(3) Let $\boldsymbol{y}$ be a homogeneous system of parameters for $S$. Then $A := \mathbb{F}[\boldsymbol{y}]$ is a *Noether normalization* for $S$, i.e., the elements $\boldsymbol{y}$ are algebraically independent over $\mathbb{F}$, and $S$ is integral over $\mathbb{F}[\boldsymbol{y}]$. Let $N$ be the largest integer with $A^N \subseteq S$, i.e., such that $S$ contains a free $A$-module of rank $N$. Then $S/A^N$ is a finitely generated $A$-torsion module, and is thus annihilated by a nonzero element $c$ of $A$.

Suppose $sy_{i+1} \in (y_1, \dots, y_i)S$ for a homogeneous element $s$ of $S$. Taking Frobenius powers, one has

$$s^q y_{i+1}^q \in (y_1^q, \dots, y_i^q)S$$

for each $q = p^e$. Since $cS \subseteq A^N$, multiplying the above by $c$ yields

$$cs^q y_{i+1}^q \in (y_1^q, \dots, y_i^q)A^N$$

for each $q = p^e$. But $\boldsymbol{y}$ is a regular sequence on the free $A$-module $A^N$, so

$$cs^q \in (y_1^q, \dots, y_i^q)A^N \subseteq (y_1^q, \dots, y_i^q)S$$

for each $q = p^e$. Hence $s \in \big((y_1, \dots, y_i)S\big)^* = (y_1, \dots, y_i)S$, where the equality is by (2).

(4) Let $z$ be a homogeneous element of $S$ having degree at least $d$. Since $z$ is integral over $A$, there exists a homogeneous equation

$$z^k + a_1 z^{k-1} + \cdots + a_k = 0$$

with $a_i \in A$. But then

$$z^N \in A + Az + \cdots + Az^{k-1}$$

for all $N \geqslant 0$. In particular, for each $q = p^e$, one has a homogeneous equation of the form

$$z^{q+k-1} = b_0 + b_1 z + \cdots + b_{k-1} z^{k-1}$$

where $b_i \in A$. Note that

$$\deg b_i \geqslant \deg b_{k-1} = \deg z^q \geqslant qd,$$

i.e., $b_i \in A_{\geqslant qd}$. This implies that

$$b_i \in (\boldsymbol{y}A)^{qd} \subseteq (y_1^q, \dots, y_d^q)A,$$

where the containment is explained by the pigeonhole principle. Consequently

$$z^{q+k-1} \in (y_1^q, \dots, y_d^q)S$$

for each $q$, so $z \in (\boldsymbol{y}S)^*$. But $(\boldsymbol{y}S)^* = \boldsymbol{y}S$ by (2). $\square$

**Exercise 1.7.** Let $Z$ denote a fixed collection of indices in an $n \times n$ matrix, and let $H_n^Z(r)$ denote the number of $n \times n$ magic squares, with line sum $r$, that have 0 in the $Z$-indices; for example, Exercise 1.3 counts magic squares with $Z := \{(1,1)\}$. Prove that $H_n^Z(r)$ is a polynomial in $r$ for each integer $r \geqslant 0$.

**Hilbert series revisited.** Recall that the Hilbert series of a standard graded ring $S$ may be written as a rational function

$$H_S(t) \;=\; \frac{h_0 + h_1 t + h_2 t^2 + \cdots + h_k t^k}{(1-t)^{\dim S}} \qquad \text{where } h_k \neq 0.$$

The coefficients of the numerator form the *h-vector* $(h_0, \dots, h_k)$ of $S$. When $S$ is Cohen-Macaulay, it is readily seen that each $h_i$ is positive:

The Hilbert series is unchanged when replacing the field $S_0 = \mathbb{F}$ by a larger field, so we may assume that $S$ has a homogeneous system of parameters $\boldsymbol{y}$ consisting of linear forms. Since $\boldsymbol{y}$ is a regular sequence on $S$, one has

$$H_S(t) \;=\; \frac{H_{S/\boldsymbol{y}S}(t)}{(1-t)^{\dim S}}.$$

But the numerator is the Hilbert series of the standard graded Artinian ring $S/\boldsymbol{y}S$, and hence is a polynomial with nonnegative coefficients. Moreover, the standard graded hypothesis shows that if $[S/\boldsymbol{y}S]_i = 0$ for some $i$, then $[S/\boldsymbol{y}S]_{\geqslant i} = 0$.

The Hilbert series of the affine semigroup rings corresponding to magic squares of size up to 6 are recorded below:

$$\begin{aligned}
\sum H_1(r)t^r \;&=\; \frac{1}{1-t},\\
\sum H_2(r)t^r \;&=\; \frac{1}{(1-t)^2},\\
\sum H_3(r)t^r \;&=\; \frac{1+t+t^2}{(1-t)^5},\\
\sum H_4(r)t^r \;&=\; \frac{1+14t+87t^2+148t^3+87t^4+14t^5+t^6}{(1-t)^{10}},\\
\sum H_5(r)t^r \;&=\; (1+103t+4306t^2+63110t^3+388615t^4+1115068t^5+1575669t^6\\
&\qquad +1115068t^7+388615t^8+63110t^9+4306t^{10}+103t^{11}+t^{12})/(1-t)^{17}\\
\sum H_6(r)t^r \;&=\; (1+694t+184015t^2+15902580t^3+567296265t^4+9816969306t^5\\
&\qquad +91422589980t^6+490333468494t^7+1583419977390t^8+3166404385990t^9\\
&\qquad +3982599815746t^{10}+3166404385990t^{11}+1583419977390t^{12}\\
&\qquad +490333468494t^{13}+91422589980t^{14}+9816969306t^{15}+567296265t^{16}\\
&\qquad +15902580t^{17}+184015t^{18}+694t^{19}+t^{20})/(1-t)^{26}
\end{aligned}$$

The Cohen-Macaulay property explains the positive coefficients in the numerators. Another immediate observation is that the $h$-vector in each case is a palindrome. This is explained by the fact that, in each case, the corresponding affine semigroup ring is Gorenstein; this comes down to the calculation of the canonical module:

By a theorem of Danilov [Da, § 4] and Stanley [St3, § 13], the canonical module of a normal affine semigroup ring is generated by monomials in the interior of the polytope; thus

$$\omega_S = x_{11}x_{12}\cdots x_{nn}\,S,$$

i.e., $\omega_S$ is generated by the monomial corresponding to the matrix in which each entry equals 1. This may be used to prove the following result, also conjectured in [ADG]:

**Theorem 1.8.** *For each $n \geqslant 1$, the polynomials $H_n(r)$ satisfy*

$$H_n(-1) \;=\; H_n(-2) \;=\; \cdots \;=\; H_n(-n+1) \;=\; 0,$$

*and*

$$H_n(r) \;=\; (-1)^{n-1} H_n(-n-r).$$

*Proof.* By the calculation of the canonical module, $S$ is Gorenstein, with $a(S) = -n$. The first statement follows from this, as does the equation

$$(-1)^{\dim S} H_S(t^{-1}) \;=\; t^n H_S(t).$$

It is a formal consequence of this that

$$H_n(-n-r) \;=\; (-1)^{\dim S - 1} H_n(r). \qquad \square$$

Yet another observation is that each $h$-vector is *unimodal*, i.e., the entries are weakly increasing, then weakly decreasing. In view of the symmetry, this can be summarized as

$$h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{\lfloor k/2 \rfloor}.$$

Following his proof of the Anand-Dumir-Gupta conjectures, Stanley [St3, p. 26] conjectured that the $h$-vector of the affine semigroup ring associated to magic squares is unimodal. This was indeed proved to be the case by Athanasiadis [At]. It thus appears natural to ask:

**Question 1.9.** Let $A$ be a standard graded Gorenstein domain in which each ideal equals its tight closure. Then, is the $h$-vector of $A$ unimodal?

Mustaţă and Payne [MP] have constructed examples of Gorenstein normal affine semigroup rings for which the $h$-vector is not unimodal, but these are not standard graded. An example of a standard graded Gorenstein ring whose $h$-vector is not unimodal may be constructed by taking a trivial extension of a Cohen-Macaulay ring by its canonical module, but then such a ring is not a domain:

**Example 1.10.** Let $X$ be a $4 \times 6$ matrix of indeterminates over a field $\mathbb{F}$. Consider the determinantal ring $R := \mathbb{F}[X]/I_4(X)$, where $I_4(X)$ is the ideal generated by the size 4 minors of the matrix $X$. Then the Hilbert series of $R$ is

$$H_R(t) \;=\; \frac{1+3t+6t^2+10t^3}{(1-t)^{21}}.$$

The canonical module $\omega$ of $R$ can be described by [BV, Theorem 8.8]: let $\mathfrak{q}$ be the prime ideal generated by the size 3 minors of the first three columns of $X$. Then $\omega = \mathfrak{q}^2$. Consider

$$S := R \oplus \omega,$$

where $\omega$ is regarded as an ideal of $S$, generated by elements of degree 1, with $\omega^2 = 0$. Then $S$ is a Gorenstein ring, with Hilbert series

$$\begin{aligned} H_S(t) &= H_R(t) + H_\omega(t) \\ &= \frac{1+3t+6t^2+10t^3}{(1-t)^{21}} + \frac{10t+6t^2+3t^3+t^4}{(1-t)^{21}} \\ &= \frac{1+13t+12t^2+13t^3+t^4}{(1-t)^{21}}. \end{aligned}$$

Evidently, the $h$-vector $(1,13,12,13,1)$ is not unimodal. ■

The unimodality results of [At] have been extended by Bruns and Römer [BR], but Question 1.9 is unresolved—to the best of our knowledge—even for Gorenstein standard graded normal affine semigroup rings.

**Linear diophantine equations.** Let $R := \mathbb{F}[x_1,\dots,x_m]$ be a polynomial ring, and $G$ a group acting *linearly* on $R$, i.e., by degree preserving $\mathbb{F}$-algebra automorphisms. Then the ring of invariant polynomials

$$R^G := \{r \in R \mid g(r) = r \text{ for all } g \in G\}$$

is a graded $\mathbb{F}$-subalgebra of $R$.

Now suppose that each of the rank 1 vector spaces $\mathbb{F}x_i$ is $G$-stable. Then each element of $G$ maps a monomial in $R$ to a scalar multiple of that monomial. It follows that if $f$ is an invariant polynomial, then each monomial that occurs in $f$ is also invariant.

**Example 1.11.** Consider the representation $G := \mathbb{F}^\times \longrightarrow \mathrm{GL}_2(\mathbb{F})$ with $\lambda \longmapsto \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{smallmatrix}\right)$, giving the action of $G$ on $R := \mathbb{F}[x,y]$ where

$$\lambda : x^i y^j \longmapsto \lambda^{i-j} x^i y^j.$$

To determine the invariant ring $R^G$, it suffices to determine the invariant monomials. Note that a monomial $x^i y^j$ is invariant precisely if $\lambda^{i-j} = 1$ for each $\lambda \in \mathbb{F}^\times$. If $\mathbb{F} = \mathbb{F}_p$, then

$$R^G \;=\; \mathbb{F}_p[x^{p-1},\, xy,\, y^{p-1}].$$

On the other hand, if $\mathbb{F}$ is an infinite field, then $\lambda^{i-j} = 1$ for each $\lambda \in G$ if and only if $i = j$. Thus, in this case,

$$R^G \;=\; \mathbb{F}_p[xy].$$

■

Let $R := \mathbb{F}[x_1,\dots,x_m]$ be a polynomial ring over a field $\mathbb{F}$, and let $G$ be the *algebraic torus* $(\mathbb{F}^\times)^\ell$, i.e., the product of $\ell$ copies of the multiplicative group of the field. Let $(h_{ij})$ be an $\ell \times m$ matrix of integers. This matrix defines an $\mathbb{F}$-linear action of $G$ on $R$ as follows:

$$(\lambda_1,\dots,\lambda_\ell)\colon x_j \longmapsto \lambda_1^{h_{1j}} \cdots \lambda_\ell^{h_{\ell j}} x_j.$$

Note that an element $(\lambda_1,\dots,\lambda_\ell)$ of $G$ maps a monomial $x_1^{b_1}\cdots x_m^{b_m}$ to the scalar multiple

$$\begin{aligned}\left(\lambda_1^{h_{11}b_1}\cdots\lambda_\ell^{h_{\ell 1}b_1}x_1^{b_1}\right)\cdots\left(\lambda_1^{h_{1m}b_m}\cdots\lambda_\ell^{h_{\ell m}b_m}x_m^{b_m}\right)& \\ = \lambda_1^{h_{11}b_1+\cdots+h_{1m}b_m}\cdots\lambda_\ell^{h_{\ell 1}b_1+\cdots+h_{\ell m}b_m}x_1^{b_1}\cdots x_m^{b_m}&.\end{aligned}$$

When $\mathbb{F}$ is an infinite field, the invariant monomials $x_1^{b_1}\cdots x_m^{b_m}$ are precisely those for which

$$\begin{pmatrix} h_{11} & \dots & h_{1m} \\ \vdots & & \vdots \\ h_{\ell 1} & \dots & h_{\ell m}\end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_m\end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix}. \tag{1.11.1}$$

Thus, the monomials that generate $R^G$ correspond precisely to the $\mathbb{N}^m$-solutions of a linear homogeneous system of equations over $\mathbb{Z}$. Consider the $\mathbb{F}$-linear map $\rho\colon R \longrightarrow R^G$ that fixes the monomials that are in $R^G$, and maps other monomials to 0; this is readily seen to be a homomorphism of $R^G$-modules that splits the inclusion $R^G \subseteq R$. Using this, it is a routine exercise that $R^G$ is a finitely generated $\mathbb{F}$-algebra, and hence that the set of $\mathbb{N}^m$-solutions to (1.11.1) is a finitely generated monoid.

Magic squares are solutions of such *linear homogeneous diophantine equations*, with the permutation matrices being a minimal set of generators for the monoid.

**Example 1.12.** Translating the five equations that spell out the equality of the line sums, it follows that a $3\times 3$ matrix $(a_{ij})$ is a magic square precisely if

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{33}\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0\end{pmatrix}.$$

As it happens, the last row is a linear combination of the previous. ■

## 2. Tight closure

We saw a first glimpse of tight closure theory in the previous section. The theory was developed by Hochster and Huneke [HH2], and has had enormous impact. It is a closure operation on ideals, first defined for rings of prime characteristic using the Frobenius map, and then extended to rings of characteristic zero by reduction mod $p$ methods. The theory leads to powerful results on unrelated topics such as rings of invariants—this is the appropriate framework for much of the previous section—integral closure of ideals and Briançon-Skoda theorems, and symbolic powers of ideals; each of these will be discussed in the coming sections.

All rings considered in these notes are assumed to be commutative, with a unit element; for the most part, they are Noetherian as well, with a few exceptions such as $R^+$, $R^\infty$, and $R^{+\text{sep}}$. Let $R$ be a ring containing a field of positive characteristic $p$. While the main case of interest is certainly the one where $R$ is an integral domain, the definition of tight closure from the previous section can be extended as follows:

**Definition 2.1.** Let $\mathfrak{a}$ be an ideal of $R$, and $z$ a ring element. Then $z$ is in $\mathfrak{a}^*$, the *tight closure* of $\mathfrak{a}$, if for each minimal prime $\mathfrak{p}$ of $R$, the image of $z$ in $R/\mathfrak{p}$ lies in $(\mathfrak{a}R/\mathfrak{p})^*$.

For an equivalent formulation, let $q$ denote a varying power of $p$, and $\mathfrak{a}^{[q]} := (a^q \mid a \in \mathfrak{a})$. Set $R^\circ$ to be the complement of the minimal primes of $R$. Then the tight closure of $\mathfrak{a}$ is

$$\mathfrak{a}^* := \left\{ z \in R \mid \text{ there exists } c \in R^\circ \text{ with } cz^q \in \mathfrak{a}^{[q]} \text{ for all } q = p^e \gg 0 \right\}.$$

**Exercise 2.2.** Verify the above equivalence, and also the following elementary facts; you may assume that the ring is Noetherian, wherever needed.

(1) The tight closure $\mathfrak{a}^*$ is indeed an ideal, and contains $\mathfrak{a}$.
(2) The tight closure of an ideal is *tightly closed*, i.e., $(\mathfrak{a}^*)^* = \mathfrak{a}^*$.
(3) The nilradical is the tight closure of the zero ideal.
(4) If $\mathfrak{a}$ is tightly closed, then so is $\mathfrak{a} : \mathfrak{b}$, for $\mathfrak{b}$ an arbitrary ideal.
(5) The intersection of an arbitrary family of tightly closed ideals is tightly closed.

**Example 2.3.** Take $R := \mathbb{F}_p[x^2, x^3]$. Then $x^3 \notin (x^2)$, though $x^3 \in (x^2)^*$ since

$$x^{3q} \ = \ x^q x^{2q} \ \in \ (x^{2q}) \qquad \text{for each } q = p^e. \qquad \blacksquare$$

**Exercise 2.4.** Let $R := \mathbb{F}_p[x,y,z]/(x^3+y^3+z^3)$. Show that

(1) $z^2 \in (x,y)^*$ and, if you like a challenge, that
(2) $xyz \in (x^2,y^2,z^2)^*$.

**Example 2.5.** Consider the hypersurface

$$\mathbb{F}_p[x,y,z,u,v,w]/(x^p - uy^p - vz^p - wxy^{p-1}z^{p-1}).$$

We claim that $x \in (y,z)^*$. To see this, verify inductively that

$$x^q \ \in \ (y^q,\ z^q,\ xy^{q-1}z^{q-1})$$

for each $q = p^e$, and choose, for example, $c := y$ as in the definition of tight closure. $\blacksquare$

**$F$-regular rings.** In Lemma 1.6 we saw that $\mathfrak{a}^* = \mathfrak{a}$ for each homogeneous ideal $\mathfrak{a}$ in a polynomial ring over an algebraically closed field. Rings of positive characteristic in which all ideals are tightly closed are *weakly $F$-regular*, while $R$ is *$F$-regular* if each localization of $R$ is weakly $F$-regular. The class of $F$-regular rings includes regular rings, determinantal rings, Plücker embeddings of Grassmannians, normal affine semigroup rings such as the magic squares rings of the previous section, and, more generally, rings of invariants of linearly reductive groups acting linearly on polynomial rings. While Brenner and Monsky [BM] have constructed striking examples demonstrating that the operation of taking the tight closure need not commute with localization, the following remains unanswered:

**Question 2.6.** Does weak $F$-regularity localize, i.e., if $R$ is a weakly $F$-regular ring, is each localization $R_{\mathfrak{p}}$ also weakly $F$-regular?

By Lyubeznik-Smith [LS], the answer is affirmative for $\mathbb{N}$-graded rings $R$ with $R_0$ a field of positive characteristic. We discuss an approach to Question 2.6 via splitting in module-finite extensions later; let us first establish that regular rings are weakly $F$-regular.

For a ring $R$ of prime characteristic $p > 0$, the map $F\colon R \longrightarrow R$ with $F(r) = r^p$ is the *Frobenius endomorphism*. Note that $R$ may be viewed as an $R$-algebra via $F$, or via any iteration of $F$. We write $F^e(R)$ to denote $R$ viewed as an $R$-algebra via the iteration $F^e$. For an ideal $\mathfrak{a} = (r_1,\dots,r_n)$ of $R$, consider the exact sequence

$$R^n \xrightarrow{(r_1\ \cdots\ r_n)} R \longrightarrow R/\mathfrak{a} \longrightarrow 0.$$

Applying $F^e(R)\otimes_R -$, the right exactness of tensor gives the exact sequence

$$R^n \xrightarrow{(r_1^q\ \cdots\ r_n^q)} R \longrightarrow F^e(R)\otimes_R R/\mathfrak{a} \longrightarrow 0,$$

where $q = p^e$, which shows that $F^e(R)\otimes_R R/\mathfrak{a} \cong R/\mathfrak{a}^{[q]}$.

If $R$ is the polynomial ring $\mathbb{F}_p[x_1,\dots,x_d]$, then $F\colon R \longrightarrow R$ may be identified with

$$\mathbb{F}_p[x_1^p,\dots,x_d^p] \subset \mathbb{F}_p[x_1,\dots,x_d]. \tag{2.6.1}$$

The monomials

$$x_d^{i_1}\cdots x_d^{i_m}$$

in which each exponent is less than $p$ form a basis for $\mathbb{F}_p[x_1,\dots,x_d]$ as an $\mathbb{F}_p[x_1^p,\dots,x_d^p]$-module, so the inclusion (2.6.1) is free, in particular, flat. More generally:

**Proposition 2.7.** *Let $R$ be a regular ring of positive prime characteristic. Then the Frobenius endomorphism of $R$ is flat.*

*Proof.* The issue is local, so assume that $R$ is a regular local ring of characteristic $p > 0$. Since $R \longrightarrow \widehat{R}$ is faithfully flat, it suffices to verify the assertion after taking the completion of $R$ at its maximal ideal. Thus, we may assume that $R = \mathbb{F}[[x_1,\dots,x_d]]$. Akin to (2.6.1), the map $F\colon R \longrightarrow R$ may be identified with

$$\mathbb{F}^p[[x_1^p,\dots,x_d^p]] \subset \mathbb{F}[[x_1,\dots,x_d]].$$

Since $\mathbb{F}$ is flat over $\mathbb{F}^p$, it follows that $\mathbb{F}[[x_1,\dots,x_d]]$ is flat over $\mathbb{F}^p[[x_1,\dots,x_d]]$. Lastly, the ring $\mathbb{F}^p[[x_1,\dots,x_d]]$ is flat over $\mathbb{F}^p[[x_1^p,\dots,x_d^p]]$ as in the polynomial case. $\square$

The converse holds by a theorem of Kunz; thus, a ring $R$ of positive prime characteristic is regular if and only if the Frobenius endomorphism $F\colon R \longrightarrow R$ is flat, see [Ku1, Her].

**Theorem 2.8.** *A regular ring of positive prime characteristic is $F$-regular.*

*Proof.* Since a regular ring is a product of domains, assume without loss of generality that the regular ring $R$ is a domain of characteristic $p > 0$. Since a localization of a regular ring is regular, it suffices to prove that $R$ is weakly $F$-regular.

The key point is the flatness of the Frobenius endomorphism, Proposition 2.7. Given an ideal $\mathfrak{a}$ of $R$ and an element $z \in R$, consider the exact sequence

$$0 \longrightarrow R/(\mathfrak{a}:z) \xrightarrow{\ z\ } R/\mathfrak{a} \longrightarrow R/(\mathfrak{a}+zR) \longrightarrow 0.$$

Tensoring with the flat $R$-module $F^e(R)$, we obtain the exact sequence

$$0 \longrightarrow R/(\mathfrak{a}:z)^{[q]} \xrightarrow{\ z^q\ } R/\mathfrak{a}^{[q]} \longrightarrow R/(\mathfrak{a}^{[q]}+z^qR) \longrightarrow 0,$$

which implies that

$$(\mathfrak{a}^{[q]}:z^q) \;=\; (\mathfrak{a}:z)^{[q]}.$$

If $z \in \mathfrak{a}^*$ then, by definition, there exists $c \neq 0$ with $cz^q \in \mathfrak{a}^{[q]}$ for all $q \gg 0$. But then

$$c \;\in\; (\mathfrak{a}^{[q]}:z^q) \;=\; (\mathfrak{a}:z)^{[q]}$$

for all $q \gg 0$. It follows that

$$c \;\in\; \bigcap_{n\geqslant 1}(\mathfrak{a}:z)^n.$$

In a Noetherian domain, the powers of a proper ideal have intersection 0, so $(\mathfrak{a}:z)$ must be the unit ideal, i.e., $z \in \mathfrak{a}$. $\square$

**Exercise 2.9.** In a regular ring of characteristic $p > 0$, prove that $(\mathfrak{a}:\mathfrak{b})^{[q]} = \mathfrak{a}^{[q]}:\mathfrak{b}^{[q]}$.

**Direct summands.** Let $S$ be a subring of a ring $R$. Then $S$ is a *direct summand of* $R$, more precisely, a direct summand of $R$ as an $S$-module, if there exists an $S$-linear map $\rho\colon R \longrightarrow S$ with $\rho(s) = s$ for all $s \in S$. In this case, there exists an $S$-module isomorphism $R \cong S \oplus M$, for $M$ an $S$-module, and for each ideal $\mathfrak{a}$ of $S$, one has

$$\mathfrak{a}R \cap S \;=\; \mathfrak{a},$$

since, if $s = \sum a_i r_i$ with $a_i \in \mathfrak{a}$ and $r_i \in R$, applying $\rho$ gives

$$s \;=\; \rho(s) \;=\; \sum a_i \rho(r_i) \;\in\; \mathfrak{a}.$$

When $R$ is a finitely generated $S$-module, the converse holds under mild hypotheses by a result of Hochster, [Ho2, Proposition 5.5]:

**Theorem 2.10.** *Let $S \subseteq R$ be a module-finite ring extension, where $S$ is a reduced excellent ring. Then $S$ is a direct summand of $R$ if (and only if) $\mathfrak{a}R \cap S = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of $S$.*

**Exercise 2.11.** This is a special case [BC, Example 1]. Let $S := \mathbb{F}_2[x,y]/(x^3,\ y^2,\ x^2y)$, and consider the module-finite extension $R := S[t]/(t^2,\ yt,\ x^2+xt)$.

Prove that $\mathfrak{a}R \cap S = \mathfrak{a}$ for each ideal $\mathfrak{a}$ of $S$, though $S$ is not a direct summand of $R$.

Now let $G$ be a group acting on a Noetherian ring $R$, and consider the ring of invariants

$$R^G := \{r \in R \mid g(r) = r \text{ for all } g \in G\}.$$

If $R^G$ is a direct summand of $R$, then $\mathfrak{a}R \cap R^G = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of $R^G$. This has several strong consequences, as we shall see. For a start, it implies that $R^G$ is a Noetherian ring: consider a chain of ideals in $R^G$,

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \dots.$$

Expanding these to ideals of $R$, we have a chain

$$\mathfrak{a}_1 R \subseteq \mathfrak{a}_2 R \subseteq \mathfrak{a}_3 R \subseteq \dots,$$

that stabilizes since $R$ is Noetherian. But $\mathfrak{a}_i R \cap R^G = \mathfrak{a}_i$, so the original chain must stabilize.

Hilbert's fourteenth problem roughly asks whether $R^G$ is Noetherian when $R$ is. The answer turns out to be negative, with the first counterexamples constructed by Nagata [Na1]. We shall say more about these issues later; for the moment, we focus on the case where $R$ is a polynomial ring over a field $\mathbb{F}$, and $G$ is a *finite* group acting on $R$ by $\mathbb{F}$-algebra automorphisms. In this case, $R^G$ is Noetherian, [AM, Exercise 7.5].

When the order $G$ is invertible in $R$, consider the *Reynolds operator* $\rho\colon R \longrightarrow R^G$ with

$$\rho(r) := \frac{1}{|G|} \sum_{g \in G} g(r). \tag{2.11.1}$$

It is easily verified that $\rho$ is an $R^G$-module homomorphism, and that $\rho(s) = s$ for all $s \in R^G$. Hence $R^G$ is a direct summand of $R$ whenever the order of $G$ is invertible in $R$.

**Example 2.12.** Consider the symmetric group $S_n$ acting on $R := \mathbb{F}[x_1, \dots, x_n]$ by permuting the variables. The invariant ring is $R^{S_n} = \mathbb{F}[e_1, \dots, e_n]$, where $e_i$ is the $i$-th elementary symmetric polynomial. Moreover, $R$ is a free $R^{S_n}$-module with basis

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \quad \text{where} \quad 0 \leqslant m_i \leqslant i-1,$$

see, for example, [Ar, Chapter II.G]. Consequently $R^{S_n}$ is a direct summand of $R$, independent of the characteristic of the field $\mathbb{F}$. ■

**Example 2.13.** Let $\mathbb{F}$ be a field of characteristic other than 2. For $n \geqslant 3$, consider the alternating subgroup $A_n$ of $S_n$ acting on $R := \mathbb{F}[x_1, \dots, x_n]$ by permuting the variables. Set

$$\Delta := \prod_{i<j} (x_i - x_j)\,.$$

Then $\sigma(\Delta) = \operatorname{sgn}(\sigma) \cdot \Delta$ for every permutation $\sigma \in S_n$, so $\Delta$ is fixed by even cycles. It is not hard to see that $R^{A_n} = \mathbb{F}[e_1, \dots, e_n, \Delta]$. Since $\Delta^2$ is fixed by all elements of $S_n$, it must be a polynomial in the elementary symmetric polynomials $e_i$, so $R^{A_n}$ is a hypersurface defined by a polynomial of the form

$$\Delta^2 - f(e_1, \dots, e_n).$$

It turns out that $R^{A_n}$ is a direct summand of $R$ if and only if $|A_n| = n!/2$ is invertible in $\mathbb{F}$. We examine the case $p = 3 = n$ here, and refer to any of [Si1, SmL, Je] for the general case. The ring of invariants is

$$R^{A_3} \;=\; \mathbb{F}[e_1, e_2, e_3, \Delta],$$

where $e_1 = \sum_i x_i$, $e_2 = \sum_{i<j} x_i x_j$, $e_3 = x_1x_2x_3$, and $\Delta = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$. Since $R^{A_3}$ is a hypersurface defined by $\Delta^2 - f(e_1, e_2, e_3)$, it follows that $\Delta \notin (e_1, e_2, e_3)R^{A_3}$. On the other hand, one may easily verify that

$$\Delta \;=\; (x_1 - x_2)(x_3e_1 + e_2) \;\in\; (e_1, e_2, e_3)R,$$

so $R^{A_3}$ is not a direct summand of $R$. ■

**Exercise 2.14.** Given that the invariant ring $R^{A_3}$ above has a presentation

$$\mathbb{F}_3[e_1, e_2, e_3, \Delta]/(\Delta^2 - e_1^2e_2^2 + e_1^3e_3 + e_2^3),$$

use the definition of tight closure to show that $\Delta \in (e_1, e_2)^*$.

It is natural to ask: for a finite group $G$, when is $R^G$ a direct summand of $R$? One answer comes from tight closure theory:

**Theorem 2.15.** *Let $R$ be a polynomial ring over a field of positive characteristic, and let $G$ be a finite group acting linearly on $R$. Then $R^G$ is a direct summand of $R$ if and only if $R^G$ is weakly $F$-regular.*

The proof uses Theorem 2.8 and basic properties of weakly $F$-regular rings:

**Theorem 2.16.** *The following hold for rings of positive characteristic:*

(1) *Direct summands of weakly $F$-regular domains are weakly $F$-regular.*
(2) *If $S \subseteq R$ is a module-finite extension of domains, then $\mathfrak{a}R \cap S \subseteq \mathfrak{a}^*$ for ideals $\mathfrak{a}$ of $S$.*
(3) *An excellent weakly $F$-regular domain is a direct summand of each module-finite extension ring.*

*Proof.* (1) Let $S$ be a direct summand of a weakly $F$-regular domain $R$. Suppose $\mathfrak{a}$ is an ideal of $S$, and $z \in \mathfrak{a}^*$. Then $cz^q \in \mathfrak{a}^{[q]}$ for all $q \gg 0$, where $c \in S$ is a nonzero element. This implies that $cz^q \in \mathfrak{a}^{[q]}R$ as well for all $q \gg 0$, so $z \in (\mathfrak{a}R)^*$. But $(\mathfrak{a}R)^* = \mathfrak{a}R$ as $R$ is weakly $F$-regular. Since $S$ is a direct summand of $R$, one has $\mathfrak{a}R \cap S = \mathfrak{a}$, so $z \in \mathfrak{a}$.

(2) Let $S \subseteq R$ be a module-finite extension of domains. At the level of fraction fields, one has a finite extension $[\operatorname{frac} R : \operatorname{frac} S] < \infty$. Choose a $\operatorname{frac} S$-linear map

$$\varphi_0 \colon \operatorname{frac} R \longrightarrow \operatorname{frac} S$$

with $\varphi_0(1) \neq 0$. Since $\varphi_0(R)$ is a finitely generated $S$-submodule of $\operatorname{frac} S$, there exists a nonzero element $d$ of $S$ with $d\varphi_0(R) \subseteq S$. Set $\varphi := d\varphi_0$, which is an $S$-linear map $R \longrightarrow S$ with $\varphi(1)$ nonzero. Set $c := \varphi(1)$.

Let $\mathfrak{a} = (x_1, \dots, x_n)$ be an ideal of $S$, and let $z \in \mathfrak{a}R \cap S$. There exist $r_i \in R$ with

$$z \;=\; r_1x_1 + \cdots + r_nx_n.$$

Taking $q = p^e$-th powers, one has

$$z^q \;=\; r_1^q x_1^q + \cdots + r_n^q x_n^q,$$

and applying $\varphi$ gives

$$cz^q \;=\; \varphi(1 \cdot z^q) \;=\; \varphi(r_1^q)x_1^q + \cdots + \varphi(r_n^q)x_n^q \;\in\; \mathfrak{a}^{[q]}$$

for all $q = p^e$, so $z \in \mathfrak{a}^*$ as desired.

(3) Let $R$ be a module-finite extension of an excellent weakly $F$-regular domain $S$. Take a minimal prime $\mathfrak{p}$ of $R$ such that $\dim R/\mathfrak{p} = \dim R$, and consider

$$S \longrightarrow R \longrightarrow R/\mathfrak{p}.$$

The composition must be injective, and a splitting of $S \longrightarrow R/\mathfrak{p}$ gives a splitting of $S \longrightarrow R$ by composition. Thus, we reduce to the case where $R$ is a domain. Let $\mathfrak{a}$ be an ideal of $S$. Then (2) gives $\mathfrak{a}R \cap S \subseteq \mathfrak{a}^*$, while $\mathfrak{a}^*$ equals $\mathfrak{a}$ since $S$ is weakly $F$-regular. Theorem 2.10 completes the proof. $\square$

*Proof of Theorem 2.15.* If $R^G$ is a direct summand of $R$, then it is weakly $F$-regular by Theorems 2.8 and 2.16 (1).

For the converse, note that for a finite group $G$ acting on $R$, the extension $R^G \subseteq R$ is integral since an element $r \in R$ is a root of the monic polynomial

$$\prod_{g \in G} (t - g(r))$$

that has coefficients in $R^G$. Hence $R^G \subseteq R$ is module-finite; now use Theorem 2.16 (3). $\square$

The following is a conjecture of Shank and Wehlau [ShW, Conjecture 1.1], reformulated using a result of Broer [Br1]. See [Br1, § 6] as well as [Br2] for more general conjectures regarding the splitting of $R^G \subseteq R$.

**Conjecture 2.17.** Let $R := \mathbb{F}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{F}$ of positive characteristic $p$. Let $G$ be a $p$-subgroup of $\mathrm{GL}_n(\mathbb{F})$, acting linearly on $R$. If $R^G$ is a direct summand of $R$, then $R^G$ is a polynomial ring.

We have proved the converse: if $R^G$ is polynomial, then it is a direct summand of $R$ by Theorems 2.8 and 2.15; alternatively, use that $R^G$ provides a homogeneous Noether normalization for $R$, and that $R$ is Cohen-Macaulay and hence free over $R^G$, Theorem 3.3.

**Splinters.** A Noetherian integral domain is a *splinter* ring if it is a direct summand of each module-finite extension domain. Splinters are normal: Suppose a fraction $a/b$ is integral over a splinter $R$. Since $R \subseteq R[a/b]$ is finite, it must be $R$-split. But then $a/b \in R$, since

$$a \;\in\; bR[a/b] \cap R \;=\; bR.$$

(1) Characteristic zero: If a normal domain $S$ contains the field of rational numbers, and $R$ is a module-finite extension domain, then the trace map of fraction fields can be used to construct a splitting

$$\frac{1}{[\operatorname{frac} R : \operatorname{frac} S]} \operatorname{Tr}_{\operatorname{frac} R/\operatorname{frac} S} \colon R \longrightarrow S.$$

Consequently, an integral domain of characteristic zero is splinter if and only if it is normal.

(2) Positive characteristic: As we saw, excellent weakly $F$-regular domains of positive characteristic are splinter, and Hochster and Huneke also proved the converse for Gorenstein rings, [HH5, Theorem 6.7]. This was later extended to the class of $\mathbb{Q}$-Gorenstein rings in [Si2], and to rings whose anti-canonical cover is Noetherian, [CEM$^+$]. There seems to be increasing evidence for the conjecture:

**Conjecture 2.18.** Let $R$ be an excellent domain containing a field of positive characteristic. Then $R$ is weakly $F$-regular if (and only if) it is splinter.

One of the incentives for proving that the splinter property and weak $F$-regularity agree for rings of positive characteristic is that it is easy to show that the localization of a splinter is splinter. It is open whether weak $F$-regularity localizes in general, Question 2.6.

The splinter property can be formulated in terms of the *plus closure* of an ideal $\mathfrak{a}$, namely

$$\mathfrak{a}^+ := \mathfrak{a}R^+ \cap R,$$

where $R^+$ denotes the integral closure of $R$ in an algebraic closure of its fraction field: by Theorem 2.10, an excellent domain $R$ is splinter if and only if $\mathfrak{a}^+ = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of $R$. By Theorem 2.16 (2), one has the containment

$$\mathfrak{a}^+ \subseteq \mathfrak{a}^*,$$

and Smith [Sm1] proved the equality $\mathfrak{a}^+ = \mathfrak{a}^*$ for parameter ideals; see Definition 3.26. On the other hand, Brenner and Monsky [BM] constructed examples where $\mathfrak{a}^+ \subsetneq \mathfrak{a}^*$, ending speculation whether equality holds in general. Note that an equivalent definition of $\mathfrak{a}^+$ is

$$\mathfrak{a}^+ = \{z \in R \mid \text{ there exists a module-finite extension domain } T \text{ with } z \in \mathfrak{a}T\}.$$

**Exercise 2.19.** Consider $\mathbb{F}_p[x,y,z]/(x^3+y^3+z^3)$ for $p \neq 3$. Show that $z^2 \in (x,y)^+$.

A study of plus closure led Hochster and Huneke to the theorem that for an excellent local domain $R$ of positive characteristic, $R^+$ is a *big Cohen-Macaulay algebra*, i.e., an $R$-algebra, not necessarily finitely generated, that is a Cohen-Macaulay $R$-module, [HH4]. A refinement of this was obtained subsequently by Huneke and Lyubeznik, see Theorem 2.27.

It turns out that $R^{+\mathrm{sep}}$, the subalgebra of *separable* elements of $R^+$, is also a big Cohen-Macaulay algebra, [Si3, SS], while the *purely inseparable* part

$$R^\infty := \bigcup R^{\frac{1}{p^e}}$$

need not be Cohen-Macaulay.

(3) Mixed characteristic: For rings of mixed characteristic, *the canonical element conjecture*, *the improved new intersection conjecture*, and *the monomial conjecture* are equivalent to the conjecture that every regular local ring is splinter, which is *the direct summand conjecture*. These and related homological questions have motivated a tremendous amount of activity including the papers [Dut, EG, Hei, Ho1, Ho3, PS, Ro1, Ro2]. The conjectures have their roots in the work of Serre [Se] and Peskine and Szpiro [PS], and grew to include conjectures due to Auslander, Bass, Hochster, and others.

In his influential CBMS lecture notes [Ho1], Hochster laid out a body of conjectures, and proved that the existence of big Cohen-Macaulay modules implies most of these. He proved that every local ring containing a field has a big Cohen-Macaulay module, thereby settling the conjectures in the equicharacteristic case. The mixed characteristic case proved more formidable: some of the conjectures including Auslander's zerodivisor conjecture and Bass's conjecture were proved by Roberts [Ro2] for rings of mixed characteristic, using local Chern characters and the intersection theory developed by Baum, Fulton, and MacPherson [BFM]; other conjectures such as the direct summand conjecture, and its equivalent formulations, remained unresolved. Heitmann [Hei] achieved the next major breakthrough, by proving these equivalent conjectures for rings of dimension up to three. Last summer, André [An1, An2] announced proofs of these conjectures, with simplifications obtained shortly after by Bhatt [Bh]. The progress comes from systematically applying Scholze's theory of perfectoid spaces [ScP]:

The homological conjectures over a ring $R$ of equicharacteristic $p > 0$ can be resolved using the Frobenius endomorphism; this was one of the major insights in the work of Peskine and Szpiro [PS] and, indeed, we saw a proof of the direct summand conjecture for rings of positive characteristic in the form of Theorems 2.8 and 2.16 (3). In mixed characteristic, while there is no Frobenius map, the theory of perfectoid spaces provides a good analog of passage to the perfection: given a ring $R$ of mixed characteristic, an extension $R \longrightarrow T$, with $T$ perfectoid, may be viewed as a substitute for the passage to the perfection $R \longrightarrow R^\infty$ in characteristic $p > 0$.

**Dagger closure.** Let $R$ be an integral domain of characteristic $p > 0$. An element $z$ belongs to the tight closure of $\mathfrak{a}$ in $R$ if, by definition, there exists a nonzero element $c$ of $R$ with

$$cz^q \in \mathfrak{a}^{[q]}$$

for each $q = p^e$. In this case, taking $q$-th roots in the above display, it follows that

$$c^{1/q}z \in \mathfrak{a}R^{1/q},$$

and hence that

$$c^{1/q}z \in \mathfrak{a}R^\infty \subseteq \mathfrak{a}R^+$$

for each $q = p^e$. Fix a valuation $v\colon R \smallsetminus \{0\} \longrightarrow \mathbb{Z}_{\geqslant 0}$, and extend to $v\colon R^+ \smallsetminus \{0\} \longrightarrow \mathbb{Q}_{\geqslant 0}$. The elements $c^{1/q}$ in $R^+$ have arbitrarily small positive valuation as $q$ varies, and multiply $z$ into the ideal $\mathfrak{a}R^+$. The surprising fact is that this characterizes tight closure, at least in a complete local domain:

**Definition 2.20.** Let $(R,\mathfrak{m})$ be a complete local domain of arbitrary characteristic. Fix a valuation $v$ that is positive on $\mathfrak{m} \smallsetminus \{0\}$, and extend to $v\colon R^+ \smallsetminus \{0\} \longrightarrow \mathbb{Q}_{\geqslant 0}$. The *dagger closure* $\mathfrak{a}^\dagger$ of an ideal $\mathfrak{a}$ is the ideal consisting of all elements $z \in R$ for which there exist elements $u \in R^+$, having arbitrarily small positive valuation, with $uz \in \mathfrak{a}R^+$.

**Theorem 2.21.** [HH3, Theorem 3.1] *Let $(R,\mathfrak{m})$ be a complete local domain of positive characteristic; fix a valuation as above. Then, for each ideal $\mathfrak{a}$ of $R$, one has $\mathfrak{a}^\dagger = \mathfrak{a}^*$.*

While tight closure is defined in characteristic zero by reduction to prime characteristic, the definition of dagger closure is characteristic-free. However, dagger closure is quite mysterious in characteristic zero and in mixed characteristic. We work out one example in characteristic zero; as this example is graded, we use the grading in lieu of a valuation, working with elements of $R^+$ that can be assigned a $\mathbb{Q}$-degree such that they satisfy a homogeneous equation of integral dependence over $R$.

**Example 2.22.** In Exercise 2.4 (1) we saw that $z^2 \in (x,y)^*$ in the hypersurface defined by $x^3+y^3+z^3$ over $\mathbb{F}_p$. In characteristic 0, one may ask if $z^2 \in (x,y)^\dagger$ over the corresponding hypersurface. This is indeed the case by [RSS, Example 2.4], as we sketch next:

Let $\theta \in \mathbb{C}$ be a primitive cube root of unity. For notational convenience, we replace the variables by scalar multiples and work instead with the hypersurface

$$R := \mathbb{C}[x,\ y,\ z]/\big(\theta x^3 + \theta^2 y^3 + z^3\big).$$

Let $R_1$ be the extension of $R$ obtained by adjoining $x_1$, $y_1$, $z_1$, where

$$x_1^3 \ = \ \theta^{1/3}x + \theta^{2/3}y, \qquad y_1^3 \ = \ \theta^{1/3}x + \theta^{5/3}y, \qquad z_1^3 \ = \ \theta^{1/3}x + \theta^{8/3}y.$$

Since $x_1$, $y_1$, and $z_1$ are cube roots of linear forms, they are assigned degree $1/3$. Note that $x$ and $y$ can be written as $\mathbb{C}$-linear combinations of $x_1^3$ and $y_1^3$, and that

$$(x_1y_1z_1)^3 \ = \ \big(\theta^{1/3}x + \theta^{2/3}y\big)\big(\theta^{1/3}x + \theta^{5/3}y\big)\big(\theta^{1/3}x + \theta^{8/3}y\big) \ = \ \theta x^3 + \theta^2 y^3 \ = \ -z^3.$$

Moreover,

$$\begin{aligned} \theta x_1^3 + \theta^2 y_1^3 + z_1^3 \ &= \ \theta\big(\theta^{1/3}x + \theta^{2/3}y\big) + \theta^2\big(\theta^{1/3}x + \theta^{5/3}y\big) + \big(\theta^{1/3}x + \theta^{8/3}y\big) \\ &= \ \big(\theta^{4/3} + \theta^{7/3} + \theta^{1/3}\big)x + \big(\theta^{5/3} + \theta^{11/3} + \theta^{8/3}\big)y \ = \ 0, \end{aligned}$$

so $R$ is a subring of the ring

$$R_1 \ = \ \mathbb{C}[x_1,\ y_1,\ z_1]/\big(\theta x_1^3 + \theta^2 y_1^3 + z_1^3\big).$$

We claim that

$$z_1 \cdot z^2 \in (x,y)R_1.$$

To see this, note that $(x_1y_1z_1)^3 = -z^3$ implies that $z = \lambda x_1y_1z_1$ for some $\lambda \in \mathbb{C}^\times$, so

$$z_1 \cdot z^2 = z_1(\lambda x_1y_1z_1)^2 \in z_1^3R_1 \subseteq (x_1^3,\, y_1^3)R_1 = (x,\, y)R_1.$$

So far, we have constructed an element $z_1$ of degree $1/3$ that multiplies $z^2$ into the ideal $(x,y)R^+$. As the notation $R_1$ might suggest, this construction can be iterated to obtain a tower of rings

$$R = R_0 \subset R_1 \subset R_2 \subset \dots \qquad \text{where} \qquad R_n = \mathbb{C}[x_n,\, y_n,\, z_n]/\big(\theta x_n^3 + \theta^2 y_n^3 + z_n^3\big),$$

and each of $x_n$, $y_n$, $z_n$ has degree $1/3^n$. ■

**Exercise 2.23.** In the notation above, show that

$$(x_n,\, y_n,\, z_n)z^2 \subseteq (x,\, y)R^+.$$

**Remark 2.24.** The rings $R_n$ are isomorphic to $R$ as abstract rings. The composition

$$R \hookrightarrow R_1 \xrightarrow{\ \cong\ } R$$

where $x_1 \longmapsto x$, and $y_1 \longmapsto y$, and $z_1 \longmapsto z$, gives an endomorphism of $R$ under which the generators of degree 1 go to elements of degree 3. The ring $R$ is the homogeneous coordinate ring of an elliptic curve, and, indeed, has several degree-increasing endomorphisms: if $E$ is an elliptic curve and $N$ a positive integer, consider the endomorphism of $E$ that takes a point $P$ to $N{\cdot}P$ under the group law. Then there exists a homogeneous coordinate ring $R$ of $E$ such that the map

$$P \longmapsto N{\cdot}P$$

corresponds to an endomorphism $\varphi\colon R \longrightarrow R$ that takes elements of degree $k$ to elements of degree $N^2k$. Arguably, Example 2.22 is atypical, in that the endomorphism exhibited takes elements of degree $k$ to elements of degree $3k$.

Extending this circle of ideas, one has the following result from [RSS]:

**Theorem 2.25.** *Let $R$ be an $\mathbb{N}$-graded domain that is finitely generated over a field $R_0$ of characteristic zero. Given a positive real number $\varepsilon$, there exists a $\mathbb{Q}$-graded finite extension domain $T$, such that the image of the induced map on local cohomology modules*

$$H^2_{\mathfrak{m}}(R)_0 \longrightarrow H^2_{\mathfrak{m}}(T)$$

*is annihilated by an element of $T$ having degree less than $\varepsilon$.*

Heitmann proof of the direct summand conjecture for rings of dimension three [Hei] has a similar flavor; he proves:

**Theorem 2.26.** *Let $(R,\mathfrak{m})$ be a local domain of dimension 3, and mixed characteristic $p$. For each $n \in \mathbb{N}$, there exists a finite extension domain $T$, such that the image of the map*

$$H^2_{\mathfrak{m}}(R) \longrightarrow H^2_{\mathfrak{m}}(T)$$

*is annihilated by $p^{1/n}$.*

Note that once a valuation $v$ on $R^+ \smallsetminus \{0\}$ is fixed, $v(p^{1/n}) = v(p)/n$ takes arbitrarily small positive values as $n$ gets large.

It is in positive characteristic that such vanishing results are the strongest; we have the theorem of Huneke and Lyubeznik [HL] mentioned earlier:

**Theorem 2.27.** *Let $(R,\mathfrak{m})$ be a local domain of positive characteristic that is a homomorphic image of a Gorenstein ring. Then there exists a finite extension domain $T$ such that the image of the induced map*

$$H^k_{\mathfrak{m}}(R) \longrightarrow H^k_{\mathfrak{m}}(T)$$

*is zero for each $k < \dim R$.*

The hypothesis of positive characteristic in the above theorem is essential. For example, let $R$ be a normal domain of characteristic zero that is not Cohen-Macaulay. If $T$ is a finite extension of $R$, then field trace provides an $R$-linear splitting of $R \hookrightarrow T$, so

$$H^k_{\mathfrak{m}}(R) \hookrightarrow H^k_{\mathfrak{m}}(T)$$

is $R$-split as well.

**Weakly $F$-regular rings revisited.** We mentioned that the splinter property and weak $F$-regularity coincide for Gorenstein rings of positive prime characteristic; moreover:

**Theorem 2.28.** *Let $(R,\mathfrak{m})$ be a Gorenstein local ring of positive prime characteristic. Fix a system of parameters $\boldsymbol{x} := x_1,\dots,x_d$ for $R$, and let $s$ be an element of $R$ that generates the socle in $R/(\boldsymbol{x})$. Then the following are equivalent:*

1. *The ring $R$ is weakly $F$-regular.*
2. *The ideal $(\boldsymbol{x})$ is tightly closed.*
3. *The element $s$ is not in $(\boldsymbol{x})^*$.*

*Proof.* The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (2) are clear. Assume (2), i.e., that the ideal $(\boldsymbol{x})$ is tightly closed. For $t \geqslant 1$, the element

$$s(x_1 \cdots x_d)^{t-1}$$

generates the socle in $R/(x_1^t,\dots,x_d^t)$. Using that $(\boldsymbol{x})^* = (\boldsymbol{x})$ and that $\boldsymbol{x}$ is a regular sequence, it is readily seen that

$$s(x_1 \cdots x_d)^{t-1} \ \notin \ (x_1^t,\dots,x_d^t)^*,$$

and hence that $(x_1^t,\dots,x_d^t)$ is tightly closed.

Let $\mathfrak{a}$ be an arbitrary $\mathfrak{m}$-primary ideal; choose an integer $t \geqslant 1$ such that $\mathfrak{a}$ contains

$$\mathfrak{b} := (x_1^t,\dots,x_d^t).$$

Since $R$ is Gorenstein, one has

$$\mathfrak{a} \;=\; \mathfrak{b} : (\mathfrak{b} : \mathfrak{a}),$$

but then Exercise 2.2 (4) implies that $\mathfrak{a}$ is tightly closed. Since $\mathfrak{m}$-primary ideals of $R$ are tightly closed, and each ideal is an intersection of $\mathfrak{m}$-primary ideals, it follows that each ideal of $R$ is tightly closed using Exercise 2.2 (5). $\square$

**Exercise 2.29.** The ring $R := \mathbb{F}[x,y]/(x^3,\ y^5)$ is Gorenstein, of dimension 0. Given the ideal $\mathfrak{a} := (x^2y,\ xy^2)$, compute $0 : \mathfrak{a}$ and $0 : (0 : \mathfrak{a})$.

Let $W$ be a multiplicative set in a ring $R$, and $\mathfrak{a}$ an ideal. It is easily seen that

$$W^{-1}(\mathfrak{a}^*) \;\subseteq\; (W^{-1}\mathfrak{a})^*,$$

and Brenner and Monsky showed that the reverse containment may fail, [BM]. However:

**Lemma 2.30.** *Let $R$ be a Noetherian ring of positive prime characteristic, and $\mathfrak{a}$ an ideal primary to a maximal ideal $\mathfrak{m}$ of $R$. Then*

$$\mathfrak{a}^* R_{\mathfrak{m}} \;=\; (\mathfrak{a}R_{\mathfrak{m}})^*.$$

*Proof.* If $z/1 \in (\mathfrak{a}R_{\mathfrak{m}})^*$, then there exists $c/1$, not in any minimal prime of $R_{\mathfrak{m}}$, with

$$cz^q/1 \;\in\; \mathfrak{a}^{[q]}R_{\mathfrak{m}} \tag{2.30.1}$$

for each $q \gg 0$. Pick an element $\delta$ that lies in precisely the minimal primes of $R$ that do not contain $c$. Then the image of $\delta$ is in each minimal prime of $R_{\mathfrak{m}}$, hence is nilpotent in $R_{\mathfrak{m}}$. Replacing $\delta$ by a power, we may assume that $\delta/1 = 0$ in $R_{\mathfrak{m}}$. Replacing $c$ by $c+\delta$, we have equation (2.30.1) with $c \in R^\circ$.

Since $\mathfrak{a}^{[q]}$ is $\mathfrak{m}$-primary, its only associated prime is $\mathfrak{m}$. But then (2.30.1) gives $cz^q \;\in\; \mathfrak{a}^{[q]}$ for each $q \gg 0$, i.e., $z \in \mathfrak{a}^*$. $\square$

As the reader has noticed, by the *associated primes* of an ideal $\mathfrak{a}$ of $R$, we mean the associated primes of $R/\mathfrak{a}$ as as $R$-module, denoted $\operatorname{Ass} R/\mathfrak{a}$. A consequence of the lemma:

**Corollary 2.31.** *Let $R$ be a Noetherian ring of positive prime characteristic. Then $R$ is weakly $F$-regular if and only if $R_{\mathfrak{m}}$ is weakly $F$-regular for each maximal ideal $\mathfrak{m}$ of $R$.*

One also has the following theorem of Murthy; see [Hu, Theorem 12.2] for a proof:

**Theorem 2.32.** *Let $R$ be a finitely generated algebra over an uncountable field of positive characteristic. Then, $R$ is weakly $F$-regular if and only if it is $F$-regular.*

**Test elements.** In the definition of tight closure, the multiplier $c$ is allowed to be any element of $R^\circ$. In several cases, there are elements $c$ that suffice for each tight closure test:

Let $R$ be a ring of prime characteristic $p > 0$. An element $c$ in $R^\circ$ is a *test element* if for each ideal $\mathfrak{a}$ of $R$, and $z \in \mathfrak{a}^*$, one has $cz^q \in \mathfrak{a}^{[q]}$ for all $q = p^e$.

**Exercise 2.33.** Verify that $c \in R^\circ$ is a test element if and only if $c\mathfrak{a}^* \subseteq \mathfrak{a}$ for each ideal $\mathfrak{a}$.

The following is a special case of [HH1, Theorem 3.4]:

**Theorem 2.34.** *Let $R$ be a reduced ring Noetherian ring of prime characteristic $p > 0$, such $R \subseteq R^{1/p}$ is module-finite. If $c$ is an element such that $R_c$ is regular, then some power of $c$ is a test element.*

The Jacobian ideal is another source of test elements; see [Ho4, § 8] for a proof of the following theorem. If $R := A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, then the Jacobian ideal $\mathcal{J}(R/A)$ may be computed as the ideal of $R$ generated by the size $n$ minors of the matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right).$$

**Theorem 2.35.** *Let $R$ be a Noetherian domain of characteristic $p > 0$ that is module-finite over a regular subring $A$. If the extension of fraction fields is separable, then each nonzero element of the Jacobian ideal $\mathcal{J}(R/A)$ is a test element.*

**Exercise 2.36.** Let $R := \mathbb{F}_p[x,y,z]/(x^3 + y^3 + z^3)$ where $p \neq 3$. Prove that $z \notin (x,y)^*$.

**Exercise 2.37.** Consider $R := \mathbb{F}_p[x,y,z]/(x^2 + y^3 + z^5)$ where $p \geqslant 7$. Then $x$ is a test element by Theorem 2.35. Use this to show that $x \notin (y,z)^*$. Conclude by Theorem 2.28 that the ring $R$ is weakly $F$-regular.

## 3. Invariant rings and the Cohen-Macaulay property

While we encountered the Cohen-Macaulay property already in the first section, let's formally record some definitions:

**Definition 3.1.** Elements $y_1, \dots, y_d$ of $R$ form a *regular sequence* on an $R$-module $M$ if

(1) $(y_1, \dots, y_d)M \neq M$, and
(2) for each $i$ with $1 \leqslant i \leqslant d$, the element $y_i$ is not a zerodivisor on $M/(y_1, \dots, y_{i-1})M$.

A local ring $(R, \mathfrak{m})$ is *Cohen-Macaulay* if some (equivalently, every) system of parameters is a regular sequence on $R$. A ring $R$ is *Cohen-Macaulay* if the local ring $R_\mathfrak{m}$ is Cohen-Macaulay for each maximal ideal $\mathfrak{m}$ of $R$.

The main case for us will be where $R$ is an $\mathbb{N}$-graded ring that is finitely generated over a field $R_0$. In this case, $R$ is Cohen-Macaulay if and only if some (equivalently, every) homogeneous system of parameters is a regular sequence. For a graded $R$-module $M$, the *depth* of $M$ is the length of a maximal sequence of homogeneous elements that form a regular sequence on $M$. Hence, $R$ is Cohen-Macaulay if and only if $\operatorname{depth} R = \dim R$.

Recall the Noether normalization theorem, in its graded form:

**Theorem 3.2.** *Let $R$ be an $\mathbb{N}$-graded ring that is finitely generated over a field $R_0 = \mathbb{F}$, and let $\boldsymbol{x} := x_1, \dots, x_d$ be a homogeneous system of parameters. Then the elements $\boldsymbol{x}$ are algebraically independent over $\mathbb{F}$, and $R$ is module-finite over its subring $\mathbb{F}[\boldsymbol{x}]$.*

This leads to another formulation of the Cohen-Macaulay property:

**Theorem 3.3.** *Let $R$ be as above, and $\boldsymbol{x}$ a homogeneous system of parameters. Then $R$ is Cohen-Macaulay if and only if it is a free module over $\mathbb{F}[\boldsymbol{x}]$.*

*Proof.* By Hilbert's syzygy theorem, $R$ has finite projective dimension over the subring

$$A := \mathbb{F}[\boldsymbol{x}].$$

The Auslander-Buchsbaum formula then says

$$\operatorname{depth} R + \operatorname{pd}_A R \ = \ \operatorname{depth} A.$$

Since $\operatorname{depth} A = \dim A = \dim R$, it follows that $\operatorname{depth} R$ equals $\dim R$ precisely if $\operatorname{pd}_A R = 0$. Since $R$ is a finitely generated graded $A$-module, it is projective if and only if it is free. $\square$

**Exercise 3.4.** Let $\mathbb{F}$ be a field. Find homogeneous systems of parameters for the rings below, and determine which are Cohen-Macaulay.

(1) $\mathbb{F}[x,\ y]/(x^2,\ xy)$
(2) $\mathbb{F}[x,\ y]/(xy)$
(3) $\mathbb{F}[x,\ y,\ z]/(xy,\ yz)$
(4) $\mathbb{F}[x,\ y,\ z]/(xy,\ yz,\ zx)$
(5) $\mathbb{F}[u,\ v,\ x,\ y]/(ux,\ uy,\ vx,\ vy)$
(6) $\mathbb{F}[x^4,\ x^3y,\ xy^3,\ y^4]$

How does the Cohen-Macaulay property arise in invariant theory? First, an example:

**Example 3.5.** Let $\mathbb{F}$ be an infinite field, and $R := \mathbb{F}[x_1, x_2, y_1, y_2]$ a polynomial ring. Consider the action of the multiplicative group $G := \mathbb{F}^\times$ on $R$, as follows:

$$\lambda \colon f(x_1,\, x_2,\, y_1,\, y_2) \longmapsto f(\lambda x_1,\, \lambda x_2,\, \lambda^{-1} y_1,\, \lambda^{-1} y_2).$$

Under this action, each monomial is mapped to a scalar multiple. Thus, $R^G$ is generated by invariant monomials, i.e., monomials $x_1^i x_2^j y_1^k y_2^l$ with

$$\lambda^{i+j-k-l} \;=\; 1 \quad \text{for all} \quad \lambda \in \mathbb{F}^\times.$$

Since $\mathbb{F}$ is infinite, the ring of invariants is

$$R^G \;=\; \mathbb{F}[x_1y_1,\, x_1y_2,\, x_2y_1,\, x_2y_2].$$

Note that $\dim R^G = 3$, for example, by examining the transcendence degree of its fraction field. The polynomial ring $S := \mathbb{F}[z_{11}, z_{12}, z_{21}, z_{22}]$ maps onto $R^G$ via the $\mathbb{F}$-algebra homomorphism $\varphi$ with $\varphi \colon z_{ij} \longmapsto x_i y_j$. It is easily seen that $\varphi(z_{11}z_{22} - z_{12}z_{21}) = 0$. Since $\dim S = 4$, the kernel of $\varphi$ must be a height one prime of $S$, and it follows that

$$\ker \varphi \;=\; (z_{11}z_{22} - z_{12}z_{21}). \qquad \blacksquare$$

Given an action of $G$ on a polynomial ring $R$, the *first fundamental problem of invariant theory*, according to Hermann Weyl [We], is to find algebra generators for the ring of invariants $R^G$, in other words to find a polynomial ring $S$ with a surjection $\varphi \colon S \longrightarrow R^G$. The *second fundamental problem* is to find relations amongst these generators, i.e., to find a free $S$-module $S^{b_1}$ that surjects onto $\ker \varphi$. In Example 3.5, we solved these two fundamental problems for the prescribed group action. In general, continuing this sequence of fundamental problems, one would like to determine the *resolution* of $R^G$ as an $S$-module, i.e., to determine an exact sequence

$$\cdots \longrightarrow S^{b_3} \longrightarrow S^{b_2} \longrightarrow S^{b_1} \longrightarrow S \xrightarrow{\varphi} R^G \longrightarrow 0. \tag{3.5.1}$$

Hilbert's syzygy theorem implies that a minimal such resolution is finite. (Since $R^G$ is a graded module over the polynomial ring $S$, *minimal* can be taken to mean that the entries of the matrices giving the maps $S^{b_{i+1}} \longrightarrow S^{b_i}$ are homogeneous nonunits; in this case, each $b_i$ is least possible.) Knowing a graded resolution, it is then easy to compute the dimension, the multiplicity and, more generally, the Hilbert series of $R^G$. Another fundamental question then arises: what is the length of the minimal resolution of $R^G$ as an $S$-module, i.e., the *projective dimension* $\operatorname{pd}_S R^G$ ? By the Auslander-Buchsbaum formula,

$$\operatorname{pd}_S R^G \;=\; \operatorname{depth} S - \operatorname{depth} R^G.$$

The polynomial ring $S$ is Cohen-Macaulay, and $\operatorname{depth} R^G \leqslant \dim R^G$, so a lower bound is

$$\operatorname{pd}_S R^G \;\geqslant\; \dim S - \dim R^G.$$

Equality holds if and only if $\operatorname{depth} R^G = \dim R^G$, i.e., precisely if $R^G$ is Cohen-Macaulay.

**Example 3.6.** As a variation of Example 3.5, let $R := \mathbb{F}[x_1, x_2, x_3, y_1, y_2]$ where $\mathbb{F}$ is infinite. Take $G := \mathbb{F}^\times$ and the action

$$\lambda \colon f(x_1,\ x_2,\ x_3,\ y_1,\ y_2) \longmapsto f(\lambda x_1,\ \lambda x_2,\ \lambda x_3,\ \lambda^{-1} y_1,\ \lambda^{-1} y_2).$$

Akin to Example 3.5, the invariant ring is

$$R^G \ = \ \mathbb{F}[x_1y_1,\ x_1y_2,\ x_2y_1,\ x_2y_2,\ x_3y_1,\ x_3y_2].$$

Using transcendence degree, one sees that $\dim R^G = 4$. The polynomial ring

$$S := \mathbb{F}[z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}]$$

maps onto $R^G$ via $z_{ij} \longmapsto x_i y_j$, and the sequence (3.5.1) takes the form

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} \Delta_1 & \Delta_2 & \Delta_3 \end{pmatrix}} S \longrightarrow R^G \longrightarrow 0.$$

where $\Delta_1 = z_{21}z_{32} - z_{31}z_{22}$, $\Delta_2 = z_{31}z_{12} - z_{11}z_{32}$, and $\Delta_3 = z_{11}z_{22} - z_{21}z_{12}$. The verification that the above sequence is exact is left to the reader. Since

$$\operatorname{pd}_S R^G \ = \ 2 \ = \ \dim S - \dim R^G,$$

it follows that $R^G$ is Cohen-Macaulay. Bearing in mind the degrees of the matrix entries, one has a graded exact sequence

$$0 \longrightarrow S^2(-3) \longrightarrow S^3(-2) \longrightarrow S \longrightarrow R^G \longrightarrow 0,$$

from which one sees that

$$H_{R^G}(t) \ = \ (1 - 3t^2 + 2t^3)H_S(t) \ = \ \frac{1 - 3t^2 + 2t^3}{(1-t)^6} \ = \ \frac{1+2t}{(1-t)^4}. \qquad \blacksquare$$

**Exercise 3.7.** Let $S$ and $T$ be algebras over a field $\mathbb{F}$, and $\varphi \colon S \longrightarrow T$ a surjective $\mathbb{F}$-algebra homomorphism. Let $\{t_i\}$ be an $\mathbb{F}$-vector space basis for $T$, and $s_i \in S$ be elements with $\varphi(s_i) = t_i$. Suppose $\mathfrak{a}$ is an ideal contained in $\ker\varphi$ such that each element of $S$ is congruent to an element in the $\mathbb{F}$-span of $\{s_i\}$ modulo $\mathfrak{a}$, prove that $\mathfrak{a} = \ker\varphi$. Use this to conclude that $S/(\Delta_1, \Delta_2, \Delta_3) \cong R^G$ in the previous example.

Yet another motivation for considering the Cohen-Macaulay property of $R^G$ comes from Theorem 3.3: when $R^G$ is Cohen-Macaulay, take a homogeneous Noether normalization

$$A := \mathbb{F}[f_1, \dots, f_d]$$

of $R^G$, and a basis for $R^G$ as an $A$-module consisting of homogeneous elements $a_1, \dots, a_m$; such a basis exists since $R^G$ is a graded free $A$-module. The decomposition

$$R^G \ = \ \bigoplus_{i=1}^{m} a_i \cdot \mathbb{F}[f_1, \dots, f_d]$$

says that each element of $R^G$ can be *uniquely* expressed in the form

$$\sum_{i=1}^{m} a_i g_i(f_1, \dots, f_d),$$

where the $g_i$ are polynomials in $d$ variables. Kempf [Ke] refers to $f_1, \dots, f_d$ as *fundamental homogeneous invariants* and to $a_1, \dots, a_r$ as *auxiliary homogeneous invariants*. Note that

$$H_{R^G}(t) \;=\; \frac{\sum_i t^{\deg a_i}}{\prod(1-t^{\deg f_j})}.$$

Hilbert described the problem of explicitly finding a set of fundamental invariants as the hardest problem of invariant theory.

We record some classical examples, pointing the reader to [We] and [DP] for details.

**Example 3.8.** Extending Examples 3.5, 3.6, let $X$ and $Y$ be $r \times n$ and $n \times s$ matrices of indeterminates over an infinite field $\mathbb{F}$, and take $R$ to be the polynomial ring in these $rn+ns$ indeterminates. Consider the action of $G := \mathrm{GL}_n(\mathbb{F})$ on $R$ where an element $M$ of $G$ maps the entries of $X$ to the corresponding entries of $XM^{-1}$ and the entries of $Y$ to those of $MY$. Then $R^G$ is the $\mathbb{F}$-algebra generated by the entries of the product matrix $XY$.

If $Z$ is an $r \times s$ matrix of new indeterminates mapping onto the entries of $XY$, the kernel of the induced $\mathbb{F}$-algebra surjection $\mathbb{F}[Z] \longrightarrow R^G$ is the ideal generated by the size $n+1$ minors of the matrix $Z$. These *determinantal rings* are the subject of [BV]. ■

**Exercise 3.9.** Let $p$ be a prime, and take $R := \mathbb{F}_p[x_1,x_2,x_3,y_1,y_2]$. Determine the invariant ring for the action of $G := \mathbb{F}_p{}^\times$, where

$$\lambda :\; f(x_1,\, x_2,\, x_3,\, y_1,\, y_2) \longmapsto f(\lambda x_1,\, \lambda x_2,\, \lambda x_3,\, \lambda^{-1} y_1,\, \lambda^{-1} y_2).$$

**Example 3.10.** Let $X$ be an $n \times d$ matrix of indeterminates over a field $\mathbb{F}$, and consider the polynomial ring $R := \mathbb{F}[X]$. Let $G := \mathrm{SL}_n(\mathbb{F})$ act on $R$, where an element $M$ of $G$ maps the $(i, j)$-th entry of the matrix $X$ to the $(i, j)$-th entry of the matrix $MX$.

Since $\det M = 1$, the size $n$ minors of $X$ are fixed by the action. When $\mathbb{F}$ is infinite, $R^G$ is the $\mathbb{F}$-algebra generated by the size $n$ minors of $X$. The ring $R^G$ is the homogeneous coordinate ring of the Grassmannian variety of $n$-dimensional subspaces of a $d$-dimensional vector space. The relations between the minors are the well-known Plücker relations.

The reader is invited to prove that $R^G$ is a unique factorization domain; see the following exercise. The key point is that since the commutator subgroup of $G = \mathrm{SL}_n(\mathbb{F})$ is $G$ itself, any homomorphism from $G$ to an abelian group must be trivial; in particular, there are no nontrivial homomorphisms from $G$ to $\mathbb{F}^\times$.

Once we know that $R^G$ is a unique factorization domain, Murthy's theorem [Mur] implies that $R^G$ is Gorenstein. More generally, the ring of invariants of a connected semisimple linear algebraic group, acting linearly on a polynomial ring, is a Cohen-Macaulay unique factorization domain, hence also Gorenstein. For more on the Gorenstein property of $R^G$ see [Wa1, Kn]. ■

**Exercise 3.11.** Let $R$ be a polynomial ring over a field $\mathbb{F}$, and $G$ a group acting on $R$ by $\mathbb{F}$-algebra automorphisms. If there are no nontrivial homomorphisms from $G$ to $\mathbb{F}^\times$, prove that $R^G$ is a unique factorization domain.

**Exercise 3.12.** Let $X$ be a $2 \times 3$ matrix of indeterminates over $\mathbb{F}$. Prove that the size 2 minors of $X$ are algebraically independent over $\mathbb{F}$. What about the minors of a $2 \times 4$ matrix?

Let us finally go through one computation in some detail:

**Example 3.13.** Let $X = (x_{ij})$ be an $n \times n$ matrix of indeterminates over an infinite field $\mathbb{F}$, and consider the polynomial ring $R := \mathbb{F}[X]$. Let $G := \mathrm{GL}_n(\mathbb{F})$ be the general linear group acting linearly on $R$, where $M$ in $G$ maps the entries of the matrix $X$ to the corresponding entries of $M^{-1}XM$. We determine the ring of invariants $R^G$. This is a special case of [Pr].

The matrices $X$ and $M^{-1}XM$ are conjugate, so the determinant and trace of $X$ and, more generally, the coefficients of its characteristic polynomial

$$p(t) := \det(tI - X)$$

are fixed by $G$. We claim that $R^G$ is the $\mathbb{F}$-algebra generated by the coefficients of $p(t)$.

Let $Y = (y_{ij})$ be an $n \times n$ matrix of new indeterminates, and set

$$S := R[Y,\ 1/\det Y].$$

Given $f(X) \in R^G$, consider $f(Y^{-1}XY) \in S$. When $Y$ is specialized to any matrix in $\mathrm{GL}_n(\mathbb{F})$, the specialization of $f(Y^{-1}XY)$ agrees with $f(X)$. Since $\mathbb{F}$ is infinite, and

$$f(Y^{-1}XY) - f(X)$$

vanishes for all specializations as above, it must vanish identically, i.e., $f(Y^{-1}XY) = f(X)$.

Let $\mathbb{K}$ be an algebraic closure of the fraction field of $R$. When we specialize the off-diagonal entries of $X$ to 0, the resulting matrix has distinct eigenvalues $x_{11}, \dots, x_{nn}$, and it follows that $X$ has distinct eigenvalues in $\mathbb{K}$. Consequently, $X$ is diagonalizable over $\mathbb{K}$, i.e., there exists a matrix $N \in \mathrm{GL}_n(\mathbb{K})$ such that $D := N^{-1}XN$ is diagonal, with the entries of $D$ being the eigenvalues of $X$. Specializing $Y$ to $N$, we see that $f(D) = f(X)$. Hence $f(X)$ is a polynomial in the entries of $D$, i.e., a polynomial in the eigenvalues of $X$.

Moreover, for a permutation $\pi \in S_n$, consider the corresponding permutation matrix $P$ in $\mathrm{GL}_n(\mathbb{K})$. Then $f(P^{-1}DP) = f(D)$, so $f(X)$ is a symmetric function of the eigenvalues of $X$. The elementary symmetric functions of the eigenvalues are, up to sign, the coefficients of the characteristic polynomial, proving the claim.

So far, we have solved the first fundamental problem for this group action. The second fundamental problem is to determine the relations, if any, between the coefficients of the characteristic polynomial of $X$. Specializing the off-diagonal entries of $X$ to 0, the coefficients of the characteristic polynomial of the resulting matrix are the $n$ elementary symmetric functions in $x_{11}, \dots, x_{nn}$ which are known to be algebraically independent. It follows that the coefficients of $p(t)$ are algebraically independent as well, so $R^G$ is isomorphic to a polynomial ring in $n$ indeterminates. ∎

In all of the examples above, $R^G$ is Cohen-Macaulay; here is one where it is not:

**Example 3.14.** Let $G := \langle \sigma \mid \sigma^2 = 1 \rangle$ act on $R := \mathbb{F}_2[x_1, x_2, x_3, y_1, y_2, y_3]$, where

$$\sigma(x_i) = y_i \quad \text{and} \quad \sigma(y_i) = x_i$$

for each $i$. Then the invariant ring is generated by the polynomials

$$x_i + y_i, \quad x_i y_i, \quad x_i y_j + x_j y_i, \quad x_1x_2x_3 + y_1y_2y_3, \qquad \text{with } 1 \leqslant i \neq j \leqslant 3$$

and $R^G$ is not Cohen-Macaulay: The elements $x_i + y_i$ and $x_i y_i$ form a homogeneous system of parameters for $R^G$, and satisfy the relation

$$(x_2y_3 + x_3y_2)(x_1 + y_1) + (x_3y_1 + x_1y_3)(x_2 + y_2) + (x_1y_2 + x_2y_1)(x_3 + y_3) \ = \ 0,$$

which is seen to be non-trivial, once one verifies that

$$(x_2y_3 + x_3y_2) \ \notin \ (x_2 + y_2,\ x_3 + y_3)R^G.$$

This is elementary, bearing in mind the degrees of the elements involved. ■

**Exercise 3.15.** Theorem 2.16 (2), applied to $R^G \subseteq R$ in the example above, says that

$$\mathfrak{a}R \cap R^G \ \subseteq \ \mathfrak{a}^*$$

for each ideal $\mathfrak{a}$ of $R^G$. Find an ideal $\mathfrak{a}$ for which $\mathfrak{a}R \cap R^G \neq \mathfrak{a}$, and verify the above.

It is no coincidence that $|G|$ is not invertible in $R$ in the above example; the following is a special case of a theorem of Hochster and Eagon, [HE, Proposition 13]:

**Theorem 3.16.** *Let $R$ be a polynomial ring over a field, and $G$ a finite group acting linearly on $R$. If $|G|$ is invertible in $R$, then $R^G$ is Cohen-Macaulay.*

*Proof.* Take a homogeneous system of parameters $\boldsymbol{y}$ for $R^G$. Since $G$ is finite, $R$ is an integral extension of $R^G$, so $\boldsymbol{y}$ is a homogeneous system of parameters for $R$ as well, and thus a regular sequence on $R$. Using the Reynolds operator, (2.11.1), it follows that $R^G$ is a direct summand of $R$ as an $R^G$-module. But then $\boldsymbol{y}$ is a regular sequence on $R^G$ as well. □

The first examples of finite groups $G$ for which $R^G$ is not Cohen-Macaulay are due to Bertin [Be], see also Fossum-Griffith [FG]. In these, $R := \mathbb{F}[x_1, \ldots, x_q]$ is a polynomial ring over a field of characteristic $p > 0$, and $q = p^e$, and $G := \mathbb{Z}/q$ acts on $R$ by a cyclic permutation of the variables. Then, for $q \geqslant 4$, the ring of invariants $R^G$ is a unique factorization domain that is not Cohen-Macaulay. Moreover, this is preserved if $R$ is replaced by its completion $\widehat{R}$ at the homogeneous maximal ideal, and the action on $\widehat{R}$ is the unique continuous action extending the one on $R$.

The proof of Theorem 3.16 works more generally to show that a direct summand $S$ of a Cohen-Macaulay ring $R$ is Cohen-Macaulay, provided that a system of parameters for $S$ forms part of a system of parameters for $R$. In general, a direct summand of a Cohen-Macaulay ring need not be Cohen-Macaulay:

**Example 3.17.** Let $\mathbb{F}$ be an infinite field of characteristic other than 3, and set

$$R := \mathbb{F}[x,y,z,s,t]/(x^3+y^3+z^3).$$

Consider the action of $G := \mathbb{F}^\times$ on $R$ where

$$\lambda\colon\, f(x,\, y,\, z,\, s,\, t) \longmapsto f(\lambda x,\, \lambda y,\, \lambda z,\, \lambda^{-1}s,\, \lambda^{-1}t).$$

Similar to Example 3.6, $R^G$ is the $\mathbb{F}$-algebra generated by $sx$, $sy$, $sz$, $tx$, $ty$, and $tz$. While $R^G$ is a direct summand of the Cohen-Macaulay ring $R$, it is not Cohen-Macaulay: the homogeneous system of parameters $sx,\ ty,\ sy-tx$ has a non-trivial relation

$$sz \cdot tz \cdot (sy-tx) \ = \ (sz)^2 \cdot ty - (tz)^2 \cdot sx. \qquad \blacksquare$$

**Reductive and linearly reductive groups.** A *linear algebraic group* is Zariski closed subgroup of a general linear group $\mathrm{GL}_n(\mathbb{F})$. A linear algebraic group $G$ is *linearly reductive* if every finite dimensional $G$-module is a direct sum of irreducible $G$-modules, equivalently, if every $G$-submodule has a $G$-stable complement. Linearly reductive groups in characteristic zero include finite groups, algebraic tori (i.e., products of copies of the multiplicative group of the field), and the classical groups $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{SL}_n(\mathbb{F})$, $\mathrm{Sp}_{2n}(\mathbb{F})$, $\mathrm{O}_n(\mathbb{F})$, and $\mathrm{SO}_n(\mathbb{F})$.

When a linearly reductive group $G$ acts linearly on a finitely generated $\mathbb{F}$-algebra $R$, there is an $R^G$-linear splitting given by the Reynolds operator

$$\rho\colon R \longrightarrow R^G.$$

One way to think of the Reynolds operator is as follows: A linear algebraic group over $\mathbb{C}$ is linearly reductive precisely if it has a Zariski dense subgroup that is a compact real Lie group; the Reynolds operator corresponds to *averaging* over the compact subgroup with respect to the Haar measure, akin to the averaging in (2.11.1). Elements that are fixed by the Zariski dense subgroup are fixed by the entire group.

A linear algebraic group is *reductive* if its largest closed connected solvable normal subgroup is an algebraic torus. In characteristic zero, linearly reductive groups are precisely those that are reductive. However, reductive groups in positive characteristic typically fail to be linearly reductive, see Example 3.23. In the preface to [Mum], Mumford conjectured that reductive groups satisfy a weaker property that should ensure that $R^G$ is Noetherian, and this led to the notion of geometrically reductive groups:

A linear algebraic group $G$ is *geometrically reductive* if for each finite dimensional $G$-module $V$, and each $G$-stable submodule $W$ of codimension one such that $G$ acts trivially on $V/W$, there exists an integer $n \geqslant 1$ such that $W \cdot \mathrm{Sym}^n V$ has a $G$-stable complement in $\mathrm{Sym}^n V$, where $\mathrm{Sym}^n V$ denotes the $n$-th symmetric power of $V$.

Nagata [Na3] proved that $R^G$ is finitely generated when $G$ is geometrically reductive, and Haboush [Hab] settled Mumford's conjecture by proving that reductive groups are geometrically reductive. It is interesting to note that for reductive groups $G$, though $\mathfrak{a}R \cap R^G$ may not be contained in $\mathfrak{a}$, one nonetheless has the following by [Na3, Lemma 5.2.B]:

$$\mathfrak{a}R \cap R^G \ \subseteq \ \operatorname{rad}\mathfrak{a}.$$

**The Hochster-Roberts Theorem.** Though the Cohen-Macaulay property is not preserved under taking direct summands, Example 3.17, the following theorem of Hochster and Roberts [HR] implies that several important invariant rings are indeed Cohen-Macaulay:

**Theorem 3.18.** *Let $G$ be a linearly reductive group over a field $\mathbb{F}$, acting linearly on a polynomial ring $R := \mathbb{F}[x_1, \dots, x_n]$. Then $R^G$ is Cohen-Macaulay.*

*More generally, a direct summand of a polynomial ring over a field is Cohen-Macaulay.*

This was extended by Hochster and Huneke to all equicharacteristic regular rings, using their construction of big Cohen-Macaulay algebras, [HH7, Theorem 2.3]: they prove that a direct summand of a regular ring containing a field is Cohen-Macaulay. More recently, André's preprint [An2] includes a construction of big Cohen-Macaulay algebras, functorial over injective maps, for local rings of mixed characteristic. This then implies that a direct summand of a regular local ring of mixed characteristic is Cohen-Macaulay.

In these notes, we will limit ourselves to proving Theorem 3.18 in the graded setting. We have seen that direct summands of weakly $F$-regular domains are weakly $F$-regular, Theorem 2.16 (1), so the positive characteristic graded case is handled by the following:

**Theorem 3.19.** *Let $R$ be an $\mathbb{N}$-graded domain that is finitely generated over a field $R_0$ of positive characteristic. Let $y_1, \dots, y_d$ be a homogeneous system of parameters for $R$. Then*

$$(y_1, \dots, y_k) : y_{k+1} \subseteq (y_1, \dots, y_k)^* \quad \text{for all} \quad 0 \leqslant k \leqslant d-1.$$

*In particular, if $R$ is weakly $F$-regular, then it is Cohen-Macaulay.*

*Proof.* We have largely seen the proof in the course of Lemma 1.6 (3), but here goes: $R$ is module-finite over $A := \mathbb{F}[y_1, \dots, y_d]$. Let $N$ be largest with $A^N \subseteq R$, in which case $R/A^N$ is annihilated by a nonzero element $c$ of $A$. If

$$sy_{k+1} \in (y_1, \dots, y_k)R,$$

applying the Frobenius endomorphism gives

$$s^q y_{k+1}^q \in (y_1^q, \dots, y_k^q)R$$

for each $q = p^e$. Multiplying the above by $c$, one then has

$$cs^q y_{k+1}^q \in (y_1^q, \dots, y_k^q)A^N.$$

But $y_1, \dots, y_{k+1}$ is a regular sequence on $A^N$, so

$$cs^q \in (y_1^q, \dots, y_k^q)A^N \subseteq (y_1^q, \dots, y_k^q)R$$

for all $q = p^e$, and hence $s \in (y_1, \dots, y_k)^*$. □

As we saw, a weakly $F$-regular is Cohen-Macaulay in the graded setting. More generally, if $R$ is a weakly $F$-regular ring that is either locally excellent, or a homomorphic image of a Cohen-Macaulay ring, then $R$ is Cohen-Macaulay; see [HH6], but after proving:

**Exercise 3.20.** Let $S \subseteq R$ be a module-finite extension of domains of positive prime characteristic. If $y_1, \dots, y_n$ is a regular sequence in $S$, show that for each $k$ one has

$$(y_1, \dots, y_k)R :_R y_{k+1} \subseteq \left((y_1, \dots, y_k)R\right)^*.$$

**Reduction modulo $p$.** We now work towards proving the Hochster-Roberts theorem for graded rings of characteristic zero. The proof will be via *reduction modulo p* methods; the technique is likely familiar from the proofs of Gauss's Lemma and Eisenstein's criterion; another great application is Dedekind's proof, from 1857, that cyclotomic polynomials are irreducible. The basic idea in Dedekind's proof, as in most reduction modulo $p$ proofs, is to start with a statement in characteristic zero, reduce modulo a prime $p$, and then exploit the Frobenius map; the technique has proved extremely useful in commutative algebra. We shall use reduction modulo $p$ methods here to prove the Hochster-Roberts theorem, and, later, the Briançon-Skoda theorem and the Ein-Lazarsfeld-Smith theorem.

There are beautiful results relating the characteristic 0 and characteristic $p$ properties of algebraic sets: Starting with a polynomial

$$f(x_1, \dots, x_d) \in \mathbb{Z}[x_1, \dots, x_d],$$

the solution set of $f = 0$ in $\mathbb{C}^d$ is a topological space. The Weil Conjectures—now theorems of Dwork, Grothendieck, and Deligne—relate the Betti numbers of this topological space to the number of roots of $f$ in finite fields $\mathbb{F}_{p^e}$. Closer to the applications that we have in mind here, is the following elementary result:

**Proposition 3.21.** *Consider a family of polynomials $f_1, \dots, f_n$ in $\mathbb{Z}[x_1, \dots, x_d]$. Then this family has a common root over $\mathbb{C}$ if and only if, for all but finitely many prime integers p, their images have a common root over $\overline{\mathbb{F}}_p$.*

*Proof.* If $(\alpha_1, \dots, \alpha_d)$ is a common root of the given polynomials in $\mathbb{C}^d$, set

$$A := \mathbb{Z}[\alpha_1, \dots, \alpha_d],$$

which is a subring of $\mathbb{C}$. Let $\mathfrak{m}$ be a maximal ideal of $A$. Then $A/\mathfrak{m}$ is a field that is finitely generated as a $\mathbb{Z}$-algebra, and is hence a finite field, see, for example, [AM, Exercise 7.6]. Let $p$ be the characteristic of $A/\mathfrak{m}$. Using $\overline{\phantom{x}}$ to denote images modulo $\mathfrak{m}$, the point

$$(\overline{\alpha}_1, \dots, \overline{\alpha}_d) \in (A/\mathfrak{m})^d$$

is a common root of the polynomials $\overline{f}_1, \dots, \overline{f}_n \in \mathbb{F}_p[x_1, \dots, x_d]$. It remains to verify that $A$ has maximal ideals containing infinitely many prime integers.

By Noether normalization, the ring

$$A_{\mathbb{Q}} := \mathbb{Q}[\alpha_1, \dots, \alpha_d]$$

is an integral extension of a polynomial subring $\mathbb{Q}[y_1, \dots, y_t]$. Each $\alpha_i$ satisfies an equation of integral dependence over $\mathbb{Q}[y_1, \dots, y_t]$. Each of these $d$ equations involves finitely many coefficients from $\mathbb{Q}$, so, after inverting a suitable integer $r$, one has an integral extension

$$\mathbb{Z}[y_1, \dots, y_t, 1/r] \subseteq A[1/r]. \tag{3.21.1}$$

For every prime integer $p$ not dividing $r$, there is a maximal ideal of

$$\mathbb{Z}[y_1,\dots,y_t,1/r]$$

that contains $p$. As the extension (3.21.1) is integral, there exists a maximal ideal of $A[1/r]$ lying over a given maximal ideal of $\mathbb{Z}[y_1,\dots,y_t,1/r]$.

Conversely, if the polynomials do not have a common root in $\mathbb{C}^d$, then Hilbert's Nullstellensatz implies that $f_1,\dots,f_n$ generate the unit ideal in $\mathbb{C}[x_1,\dots,x_d]$, i.e., that

$$\mathbb{C}[x_1,\dots,x_d]/(f_1,\dots,f_n) \;=\; 0.$$

But $\mathbb{C}$ is faithfully flat over $\mathbb{Q}$, so

$$\mathbb{Q}[x_1,\dots,x_d]/(f_1,\dots,f_n) \;=\; 0$$

as well, i.e., $f_1,\dots,f_n$ generate the unit ideal in $\mathbb{Q}[x_1,\dots,x_d]$. Taking an equation and clearing denominators, one has

$$f_1g_1+\cdots+f_ng_n \;=\; m,$$

where $g_1,\dots,g_n \in \mathbb{Z}[x_1,\dots,x_d]$, and $m$ is a nonzero integer. For each prime integer $p$ not dividing $m$, the images of $f_1,\dots,f_n$ generate the unit ideal in $\mathbb{F}_p[x_1,\dots,x_d]$, and hence cannot have a common root over $\overline{\mathbb{F}}_p$. $\square$

We now return to the Hochster-Roberts theorem in the following form:

**Theorem 3.22.** *Let $R$ be an $\mathbb{N}$-graded polynomial ring over a field $R_0$ of characteristic* 0*, and $S$ a graded $R_0$-subalgebra that is a direct summand of $R$. Then $S$ is Cohen-Macaulay.*

*Proof.* Let $R = \mathbb{F}[x_1,\dots,x_n]$, where $x_i$ are indeterminates. Note that $S$ must be a finitely generated $\mathbb{F}$-algebra, say $S = \mathbb{F}[u_1,\dots,u_m]$. Let $y_1,\dots,y_d$ be a homogeneous system of parameters for $S$. If $S$ is not Cohen-Macaulay, then there exist homogeneous $s_i \in S$ with

$$s_1y_1+\cdots+s_{k+1}y_{k+1} \;=\; 0 \tag{3.22.1}$$

and $s_{k+1} \notin (y_1,\dots,y_k)S$. Since $S$ is a direct summand of $R$, it follows that

$$s_{k+1} \;\notin\; (y_1,\dots,y_k)R. \tag{3.22.2}$$

Note that one has the containments

$$\mathbb{F}[y_1,\dots,y_d] \;\subseteq\; \mathbb{F}[u_1,\dots,u_m] \;\subseteq\; \mathbb{F}[x_1,\dots,x_n].$$

The elements $y_1,\dots,y_d$ are algebraically independent over $\mathbb{F}$; enlarge this to a transcendence basis $y_1,\dots,y_n$ for $\mathbb{F}(x_1,\dots,x_n)$ over $\mathbb{F}$, such that each $y_i \in \mathbb{F}[x_1,\dots,x_n]$.

Let $A$ be a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{F}$ such that

(i) one has $A[y_1,\dots,y_d] \subseteq A[u_1,\dots,u_m] \subseteq A[x_1,\dots,x_n]$, i.e., such that the ring $A$ contains coefficients needed to express $u_1,\dots,u_m$ and $y_1,\dots,y_d$ as polynomials in $x_1,\dots,x_n$, and to express $y_1,\dots,y_d$ as polynomials in $u_1,\dots,u_m$,

(ii) the extension $A[y_1,\dots,y_d] \subseteq A[u_1,\dots,u_m]$ is integral,

(iii) each $s_i \in A[u_1,\dots,u_m]$,

(iv) each $y_1,\dots,y_n$ is an element of $A[x_1,\dots,x_n]$,

(v) each $x_i$ is algebraically dependent over $(\operatorname{frac} A)(y_1,\dots,y_n)$; moreover, for each $x_i$, take its minimal polynomial over the field $(\operatorname{frac} A)(y_1,\dots,y_n)$ and invert the nonzero elements of $A$ that occur as coefficients in these $n$ polynomials.

The last step ensures that if $\mathfrak{m}$ is a maximal ideal of $A$, then the indeterminates $x_1,\dots,x_n$ are algebraically dependent over $A/\mathfrak{m}[y_1,\dots,y_n]$. It then follows that the images of $y_1,\dots,y_n$, and hence of $y_1,\dots,y_d$, remain algebraically independent over $A/\mathfrak{m}$.

Let $D = \deg s_{k+1}$. In view of (3.22.2), one has an inclusion of finite rank $\mathbb{F}$-vector spaces

$$\mathbb{F}\cdot s_{k+1} \hookrightarrow \big[R/(y_1,\dots,y_k)\big]_D.$$

This restricts to an inclusion of finitely generated $A$-modules

$$A\cdot s_{k+1} \hookrightarrow \big[A[x_1,\dots,x_n]/(y_1,\dots,y_k)\big]_D. \tag{3.22.3}$$

Enlarge $A$ by inverting a nonzero element such that the above inclusion then becomes a split inclusion of free $A$-modules.

Let $\mathfrak{m}$ be a maximal ideal of $A$. Then $\kappa := A/\mathfrak{m}$ is a field that is finitely generated as a $\mathbb{Z}$-algebra, hence is a finite field. Set

$$R_\kappa := \kappa[x_1,\dots,x_n] \quad \text{and} \quad S_\kappa := \kappa[u_1,\dots,u_m].$$

Note that we have

$$\kappa[y_1,\dots,y_d] \subseteq S_\kappa \subseteq R_\kappa,$$

where $\kappa[x_1,\dots,x_n]$ and $\kappa[y_1,\dots,y_d]$ are polynomial rings over the field $\kappa$, and the extension $\kappa[y_1,\dots,y_d] \subseteq \kappa[u_1,\dots,u_m]$ is module-finite. In particular, $y_1,\dots,y_d$ is a homogeneous system of parameters for $\kappa[u_1,\dots,u_m]$.

By (3.22.1), one has

$$s_{k+1}y_{k+1} \in (y_1,\dots,y_k)S_\kappa,$$

so Theorem 3.19 implies that

$$s_{k+1} \in \big((y_1,\dots,y_k)S_\kappa\big)^*.$$

But then

$$s_{k+1} \in \big((y_1,\dots,y_k)R_\kappa\big)^* = (y_1,\dots,y_k)R_\kappa.$$

On the other hand, applying $-\otimes_A \kappa$ to (3.22.3) gives

$$\kappa\cdot s_{k+1} \hookrightarrow \big[R_\kappa/(y_1,\dots,y_k)\big]_D,$$

a contradiction. $\square$

One subtle point in the proof above: the property that $S$ is a direct summand of $R$ may not be preserved when we reduce modulo primes $p$, as we see in the following example:

**Example 3.23.** Let $\mathbb{F}$ be an infinite field, and consider $G := \mathrm{SL}_2(\mathbb{F})$ acting on the polynomial ring $R := \mathbb{F}[u,v,w,x,y,z]$, where an element $M$ of $G$ maps the entries of the matrix

$$X := \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}$$

to those of the matrix $MX$. This is a special case of Example 3.10; the ring of invariants is $R^G = \mathbb{F}[\Delta_1, \Delta_2, \Delta_3]$, where

$$\Delta_1 := vz - wy, \quad \Delta_2 := wx - uz, \quad \Delta_3 := uy - vx$$

are the size 2 minors of the matrix $X$. These minors are algebraically independent over $\mathbb{F}$ by Exercise 3.12, and hence $R^G$ is a polynomial ring. When $\mathbb{F}$ has characteristic zero, the group $G$ is linearly reductive, and hence $R^G$ is a direct summand of $R$. We shall see that $R^G$ is not a direct summand of $R$ when $\mathbb{F}$ has positive characteristic $p$:

Let $\mathfrak{a} := (\Delta_1, \Delta_2, \Delta_3)R$, and consider the local cohomology module

$$H^3_{\mathfrak{a}}(R) := \varinjlim_t \mathrm{Ext}^3_R(R/\mathfrak{a}^t,\ R) \ \cong\ \varinjlim_{q=p^e} \mathrm{Ext}^3_R(R/\mathfrak{a}^{[q]},\ R),$$

where the isomorphism holds since the sequences of ideals $\{\mathfrak{a}_t\}_{t\in\mathbb{N}}$ and $\{\mathfrak{a}^{[p^e]}\}_{e\in\mathbb{N}}$ are cofinal. Since $R/\mathfrak{a}$ is Cohen-Macaulay, the Auslander-Buchsbaum formula gives

$$\mathrm{pd}_R R/\mathfrak{a} = 2.$$

Tensoring a projective resolution of $R/\mathfrak{a}$ with $F^e(R)$ gives a projective resolution of $R/\mathfrak{a}^{[q]}$ by the flatness of the Frobenius endomorphism. Thus $\mathrm{pd}_R R/\mathfrak{a}^{[q]} = 2$, so

$$\mathrm{Ext}^3_R(R/\mathfrak{a}^{[q]},\ R) \ =\ 0$$

for all $q = p^e$, and hence $H^3_{\mathfrak{a}}(R) = 0$. Using the alternative description of $H^\bullet_{\mathfrak{a}}(R)$ as the cohomology of the Čech complex

$$0 \longrightarrow R \longrightarrow R_{\Delta_1} \oplus R_{\Delta_2} \oplus R_{\Delta_3} \longrightarrow R_{\Delta_2\Delta_3} \oplus R_{\Delta_3\Delta_1} \oplus R_{\Delta_1\Delta_2} \longrightarrow R_{\Delta_1\Delta_2\Delta_3} \longrightarrow 0,$$

the surjectivity of the penultimate map implies that

$$\frac{1}{\Delta_1\Delta_2\Delta_3} \ =\ \frac{r_1}{(\Delta_2\Delta_3)^t} + \frac{r_2}{(\Delta_3\Delta_1)^t} + \frac{r_3}{(\Delta_1\Delta_2)^t}$$

for some $r_i \in R$ and $t \in \mathbb{N}$. Multiplying by $(\Delta_1\Delta_2\Delta_3)^t$, one has

$$(\Delta_1\Delta_2\Delta_3)^{t-1} \ =\ r_1\Delta_1^t + r_2\Delta_2^t + r_3\Delta_3^t \ \in\ (\Delta_1^t, \Delta_2^t, \Delta_3^t)R.$$

Since $(\Delta_1\Delta_2\Delta_3)^{t-1} \notin (\Delta_1^t, \Delta_2^t, \Delta_3^t)R^G$, it follows that $R^G$ is not a direct summand of $R$. ■

We have seen that weak $F$-regularity is preserved under taking direct summands and that, under mild hypotheses, it implies the Cohen-Macaulay property. Here is a class of rings of characteristic zero with these properties:

**Definition 3.24.** Let $X$ be an irreducible normal variety over an algebraically closed field of characteristic zero. Then $X$ has *rational singularities* if for some (equivalently, every) desingularization $f\colon Z \longrightarrow X$, one has

$$R^i f_*(\mathcal{O}_Z) \;=\; 0$$

for all $i \geqslant 1$. If $X$ has rational singularities, then the local rings of $X$ are Cohen-Macaulay.

We say that $R$ has rational singularities if $\operatorname{Spec} R$ has rational singularities. There are numerical criteria that detect when a graded ring has rational singularities due to Flenner [Fl] and Watanabe [Wa2]. It is a theorem of Boutot [Bo] that the property of having rational singularities is preserved under taking direct summands; see also Gurjar [Gu]:

**Theorem 3.25.** *Let $R$ be a finitely generated algebra over a field of characteristic zero, and let $S$ be a direct summand of $R$. If $R$ has rational singularities, then so does $S$.*

Rational singularities are related to a property that arises in tight closure theory:

**Definition 3.26.** An ideal $\mathfrak{a} := \boldsymbol{x}R$ is a *parameter ideal* if the images of $\boldsymbol{x}$ form part of a system of parameters of $R_{\mathfrak{p}}$, for each prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$. A ring $R$ of positive prime characteristic is *$F$-rational* if each parameter ideal is tightly closed.

While it is open whether a localization of a weakly $F$-regular is weakly $F$-regular, the situation is better with $F$-rationality; proofs of the following may be found in [HH6]:

**Theorem 3.27.** *The following hold for rings of prime characteristic:*

(1) *An $F$-rational ring is normal.*
(2) *A Gorenstein ring is $F$-rational if and only if it is weakly $F$-regular.*

*Suppose, in addition, that $R$ is a homomorphic image of a Cohen-Macaulay ring. Then:*

(3) *If $R$ is $F$-rational, then it is Cohen-Macaulay.*
(4) *If $R$ is $F$-rational, then so is each localization of $R$.*
(5) *If $R$ is local, then it is $F$-rational if and only if it is equidimensional and the ideal generated by one system of parameters is tightly closed.*

**Definition 3.28.** Let $R$ be a finitely generated algebra over a field $\mathbb{F}$ of characteristic zero. Then $R$ is of *$F$-rational type* (or *$F$-regular type*) if there exists a finitely generated $\mathbb{Z}$-subalgebra $A$ of $\mathbb{F}$, and a finitely generated free $A$-algebra $R_A$ with

$$R_A \otimes_A \mathbb{F} \;\cong\; R,$$

such that for all maximal ideals $\mathfrak{m}$ in a Zariski dense open subset of $\operatorname{Spec} A$, the rings

$$R_A \otimes_A A/\mathfrak{m}$$

are $F$-rational (or $F$-regular).

Smith [Sm2] proved that a ring of $F$-rational type has rational singularities, and the converse was proved by Hara [Har] as well as Mehta-Srinivas [MeS]. Combining these:

**Theorem 3.29.** *Let $R$ be a ring finitely generated over a field of characteristic* 0*. Then $R$ has rational singularities if and only if it is of $F$-rational type.*

For other striking connections between tight closure theory and singularities in characteristic zero, see [HW]. The following example from [KSS$^+$] includes rings that are of $F$-rational type, but not of $F$-regular type; see [Wa3] for examples in dimension 2.

**Example 3.30.** Let $\mathbb{F}$ be a field, let $m,n$ be integers with $m,n \geqslant 2$, and let

$$\mathcal{R} := \mathbb{F}[x_1,\dots,x_m,y_1,\dots,y_n]/(f)$$

be a normal $\mathbb{N}^2$-graded hypersurface where $\deg x_i = (1,0)$, $\deg y_j = (0,1)$. Suppose $f$ is homogeneous of degree $(d,e)$, for $d,e$ positive. Set

$$\mathcal{R}_\Delta := \bigoplus_{i\geqslant 0} \mathcal{R}_{i,i}.$$

Then the ring $\mathcal{R}_\Delta$ is Cohen-Macaulay if and only if $d-m<e$ and $e-n<d$, and $\mathcal{R}_\Delta$ is Gorenstein if and only if $d-m=e-n$.

Suppose $\mathbb{F}$ has characteristic zero, and $f$ is a generic polynomial of degree $(d,e)$. Then $\mathcal{R}_\Delta$ if of $F$-rational type if and only if it is Cohen-Macaulay and $d<m$ or $e<n$, while $\mathcal{R}_\Delta$ is of $F$-regular type if and only if $d<m$ and $e<n$. ■

**Definition 3.31.** Let $R$ be a ring of prime characteristic $p>0$. Then $R$ is *$F$-pure* if the Frobenius map $F\colon R \longrightarrow R$ is *pure*, i.e., if

$$F\otimes 1\colon R\otimes_R M \longrightarrow R\otimes_R M$$

is injective for each $R$-module $M$.

Taking $M := R/\mathfrak{a}$, the map displayed above takes the form

$$R/\mathfrak{a} \xrightarrow{\ r\longmapsto r^p\ } R/\mathfrak{a}^{[p]}.$$

It follows that if $R$ is $F$-pure, then for each ring element $r$ and ideal $\mathfrak{a}$, one has $r^p \in \mathfrak{a}^{[p]}$ if and only if $r\in\mathfrak{a}$. The converse holds as well when $R$ is Noetherian and $F\colon R\longrightarrow R$ is a finite map. (Exercise: Why?)

The following is referred to as *Fedder's criterion*; see [Fe] for a proof:

**Theorem 3.32.** *Let $(R,\mathfrak{m})$ be a regular local ring of characteristic $p>0$, and $\mathfrak{a}$ an ideal. Then the ring $R/\mathfrak{a}$ is $F$-pure if and only if for some (equivalently, all) $q=p^e$, one has*

$$(\mathfrak{a}^{[q]} : \mathfrak{a}) \not\subseteq \mathfrak{m}^{[q]}.$$

**Exercise 3.33.** Determine the primes $p$ for which $\mathbb{F}_p[x,y,z]/(x^3+y^3+z^3)$ is $F$-pure.

**Exercise 3.34.** Determine the primes $p$ for which $\mathbb{F}_p[x,y,z]/(x^2+y^3+z^6)$ is $F$-pure.

**Exercise 3.35.** If $R$ is a polynomial ring over a field of positive characteristic, and $\mathfrak{a}$ an ideal generated by square-free monomials in the indeterminates, prove that $R/\mathfrak{a}$ is $F$-pure.

## 4. The Briançon-Skoda Theorem

**Definition 4.1.** Let $\mathfrak{a}$ be an ideal of a ring $R$. An element $z$ is in the *integral closure* of $\mathfrak{a}$, denoted $\overline{\mathfrak{a}}$, if it satisfies an equation of the form

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \;=\; 0,$$

with $a_k \in \mathfrak{a}^k$ for each $k$. When $R$ is Noetherian, an equivalent definition is

$$\overline{\mathfrak{a}} := \Big\{ z \in R \mid \text{ there exists } c \in R^\circ \text{ with } cz^k \in \mathfrak{a}^k \text{ for all } k \gg 0 \Big\},$$

alternatively, one may require the condition above for infinitely many $k \in \mathbb{N}$. Also, in a Noetherian ring, $z \in \overline{\mathfrak{a}}$ if and only if $\varphi(z) \in \mathfrak{a}V$ for each homomorphism $\varphi \colon R \longrightarrow V$, with $V$ a discrete valuation ring. Lastly, when $R$ is a Noetherian domain, one has

$$\overline{\mathfrak{a}} \;=\; \bigcap \mathfrak{a}V \cap R,$$

where the intersection is over discrete valuation rings $V$, between $R$ and $\operatorname{frac} R$, for which the maximal ideal of $V$ contracts to a maximal ideal of $R$, see [SH, Proposition 6.8.4].

It is easy to see that $\overline{\mathfrak{a}}$ is an ideal of $R$, and that

$$\mathfrak{a} \;\subseteq\; \overline{\mathfrak{a}} \;\subseteq\; \operatorname{rad} \mathfrak{a}.$$

Moreover, if $R$ is Noetherian and of characteristic $p > 0$, one of the characterizations gives

$$\mathfrak{a}^* \;\subseteq\; \overline{\mathfrak{a}}.$$

Taking the ideal $(x^2, y^2)$ in $\mathbb{F}[x, y]$, one sees that tight closure is tighter.

**Exercise 4.2.** Let $\mathfrak{a}$ be an ideal of a ring $R$. Prove that $z \in \operatorname{rad} \mathfrak{a}$ if and only if $\varphi(z) \in \mathfrak{a}\mathbb{F}$ for each homomorphism $\varphi \colon R \longrightarrow \mathbb{F}$, with $\mathbb{F}$ a field.

**Exercise 4.3.** Let $(R, \mathfrak{m})$ be a local domain. Prove that every integrally closed ideal of $R$ is the intersection of $\mathfrak{m}$-primary integrally closed ideals.

**Proposition 4.4.** *Let $R := \mathbb{C}\{x_1, \ldots, x_d\}$ be the ring of power series in $d$ variables, with complex coefficients, that are convergent in a neighborhood of the origin. If $f$ belongs to the maximal ideal of $R$, then*

$$f \;\in\; \overline{\left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d} \right)}.$$

*Proof.* Set $\mathfrak{a}$ to be the ideal generated by all the partial derivatives $\partial f / \partial x_i$, and suppose that $f \notin \overline{\mathfrak{a}}$. Then there exists a discrete valuation ring $(V, tV)$, with $R \subseteq V \subseteq \operatorname{frac} R$, such that the maximal ideal of $V$ contracts to that of $R$, and $f \notin \mathfrak{a}V$. Note that $f \in tV$ and that $\mathfrak{a}V = t^m V$ for some integer $m \geqslant 1$. The completion of $V$ is a formal power series ring $\widehat{V} = \mathbb{F}[[t]]$, for $\mathbb{F}$ a field containing $\mathbb{C}$. Working in $\widehat{V}$, the chain rule gives

$$\frac{df}{dt} \;=\; \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} \, \frac{\partial x_i}{\partial t},$$

which shows that $df/dt$ belongs to the ideal $\mathfrak{a}\widehat{V}$. Since $f$ is a power series belonging to the ideal $t\widehat{V}$, with $df/dt \in t^m \widehat{V}$, it follows that $f \in t^{m+1}\widehat{V}$, a contradiction. $\square$

The proposition implies that *some* power $f^k$ of $f$ belongs to the ideal generated by the partial derivatives; Mather asked if there is a bound on this power $k$, [Wal]. If $f$ is a homogeneous polynomial, then $k = 1$ suffices, since the Euler identity gives

$$(\deg f)f \;=\; \left(x_1\frac{\partial f}{\partial x_1} + \cdots + x_d\frac{\partial f}{\partial x_d}\right).$$

However, $k = 1$ may not be sufficient for an inhomogeneous polynomial:

**Exercise 4.5.** Take $f = x^2y^2 + x^5 + y^5$ in $\mathbb{C}\{x,y\}$. Determine the smallest power $k$ with

$$f^k \;\in\; \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

**Exercise 4.6.** Take $f = x^2 + y^3 + z^6 + xyz$ in $\mathbb{C}\{x,y\}$. Determine the smallest power $k$ with

$$f^k \;\in\; \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Spoiler alert! By a result of Saito [Sa], for $f$ defining a hypersurface with an isolated singular point at the origin in $\mathbb{C}^d$, one has $f \in (\partial f/\partial x_1, \ldots, \partial f/\partial x_d)$ precisely when, after a change of coordinates, $f$ can be represented as a quasihomogeneous polynomial.

Briançon and Skoda [SB] answered Mather's question by proving that $f^d$ belongs to the ideal generated by the partial derivatives; as they say, this is *meilleur possible*:

**Example 4.7.** In the ring $\mathbb{C}\{x_1,\ldots,x_d\}$, consider

$$f \;=\; (x_1\cdots x_d)^3 + x_1^{3d-1} + \cdots + x_d^{3d-1}.$$

Then

$$f^{d-1} \;\notin\; \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\right).$$

■

The proof by Briançon and Skoda uses plurisubharmonic functions, the convergence of integrals, and a hard theorem of Skoda. The absence of a purely algebraic proof for such an algebraic result was highlighted by Hochster in his 1979 lecture series *Analytic methods in commutative algebra* at George Mason University, and became, to quote Lipman and Teissier, *something of a scandal—perhaps even an insult—and certainly a challenge*. The first algebraic proofs were found by Lipman and Teissier [LiT]; subsequently the result was extended to ideals in arbitrary regular rings by Lipman and Sathaye [LiS]:

**Theorem 4.8.** *If $\mathfrak{a}$ is an ideal generated by $n$ elements in a regular ring, then $\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}$.*

Hochster and Huneke [HH2] gave an elementary tight closure proof of the following:

**Theorem 4.9.** *Let $R$ be a Noetherian ring of positive prime characteristic, and $\mathfrak{a}$ an ideal generated by $n$ elements. Then*

$$\overline{\mathfrak{a}^n} \;\subseteq\; \mathfrak{a}^*.$$

*In particular, if $R$ is weakly $F$-regular, then $\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}$.*

*Proof.* It suffices to verify the assertion modulo each minimal prime; assume $R$ is a domain. Let $\mathfrak{a} = (x_1, \dots, x_n)$ and $z \in \overline{\mathfrak{a}^n}$. By one of the characterizations of integral closure, there is a nonzero element $c$ in $R$ such that

$$cz^k \in \mathfrak{a}^{nk}$$

for all $k \gg 0$. By the pigeonhole principle,

$$\mathfrak{a}^{nk} \subseteq (x_1^k, \dots, x_n^k),$$

so restricting $k$ to $q = p^e$, we get $cz^q \in \mathfrak{a}^{[q]}$ for $q \gg 0$, and hence that $z \in \mathfrak{a}^*$. □

**Corollary 4.10.** *Let $R$ be a Noetherian ring of positive prime characteristic. If $\mathfrak{a}$ is a principal ideal, then $\mathfrak{a}^* = \overline{\mathfrak{a}}$. If $R$ is weakly $F$-regular, then it is normal.*

*Proof.* The statement regarding principal ideals is immediate from the theorem. Suppose $R$ is weakly $F$-regular. Then $0^* = 0$ implies that $R$ is reduced. If $a/b$ is an element of the total ring of fractions that is integral over $R$, then $a \in \overline{bR} = bR$, so $a/b \in R$. □

**Exercise 4.11.** Consider $\mathfrak{a} := (x, y)$ in $\mathbb{F}_p[x, y, z]/(x^3 + y^3 + z^3)$. Compute $\overline{\mathfrak{a}^2}$, and verify that it is indeed contained in $\mathfrak{a}^*$.

**Exercise 4.12.** For $R$ and $\mathfrak{a}$ as in Theorem 4.9, prove that

$$\overline{\mathfrak{a}^{m+n}} \subseteq \left(\mathfrak{a}^{m+1}\right)^*$$

for each $m \geqslant 0$. If particular, if $R$ is weakly $F$-regular, then $\overline{\mathfrak{a}^{m+n}} \subseteq \mathfrak{a}^{m+1}$.

Here is an extension due to Aberbach and Huneke, [AH3]:

**Theorem 4.13.** *Let $R$ be an $F$-rational ring of positive characteristic, or a finitely generated algebra over a field of characteristic zero such that $R$ has rational singularities. If $\mathfrak{a}$ is an $n$-generated ideal of $R$, then, for all $m \geqslant 0$, one has*

$$\overline{\mathfrak{a}^{m+n}} \subseteq \mathfrak{a}^{m+1}.$$

In another direction, Lipman [Lip] used adjoint ideals to obtain improved Briançon-Skoda theorems; adjoint ideals are now more popularly known as *multiplier ideals*, see the survey [BL]. Other improvements involving coefficient ideals may be found in [AH2], and for applications to Rees rings, see [AH1, AHT]. Rees and Sally extended the Briançon-Skoda theorem from another viewpoint in [RS], and Swanson's related work on joint reductions appears in [Sw1, Sw2]. In Wall's lectures on Mather's work, where it all began, it is amusing to find the sentence ([Wal, page 185]):

*Once the seed of algebra is sown, it grows fast.*

We next prove the Briançon-Skoda theorem for regular rings of characteristic zero; the extension $\overline{\mathfrak{a}^{m+n}} \subseteq \mathfrak{a}^{m+1}$ proceeds similarly. We will use the following lemma; a proof may be found in [Ma1, Chapter 8].

**Lemma 4.14** (Generic freeness)**.** *Let $A$ be a Noetherian domain, and $R$ a finitely generated $A$-algebra. Let $M$ be a finitely generated $R$-module. Then there exists a nonzero element $a$ in $A$ such that $M_a$ is a free $A_a$-module.*

It is worth emphasizing that when the lemma is applied to a domain $A$, finitely generated over the integers, the extension $A_a$ is also a finitely generated $\mathbb{Z}$-algebra.

**Theorem 4.15.** *Let $\mathfrak{a}$ be an $n$-generated ideal in a regular ring that contains a field of characteristic zero. Then*

$$\overline{\mathfrak{a}^n} \subseteq \mathfrak{a}.$$

The proof is easiest for polynomials rings; we present this first:

*Proof of Theorem 4.15 in the polynomial case.* Suppose $R := \mathbb{F}[x_1,\dots,x_d]$ is a polynomial ring over a field $\mathbb{F}$ of characteristic 0. Fix generators $\boldsymbol{f} := f_1,\dots,f_n$ for the ideal $\mathfrak{a}$. If the theorem is false, then there exists an element $z$ in $\overline{\mathfrak{a}^n} \smallsetminus \mathfrak{a}$. Choose $a_k \in (\boldsymbol{f})^{nk}$ with

$$z^l + a_1 z^{l-1} + \cdots + a_l \;=\; 0. \tag{4.15.1}$$

Let $A$ be a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{F}$ containing the coefficients of $z, f_i$, and $a_i$ as polynomials in $\boldsymbol{x} := x_1,\dots,x_d$, and also the coefficients of polynomials needed to express each $a_k$ as an element of $(\boldsymbol{f})^{nk}$. Thus, we have ensured that equation (4.15.1) holds in $A[\boldsymbol{x}]$.

Consider the exact sequence

$$0 \longrightarrow zA[\boldsymbol{x}]/(\boldsymbol{f}) \longrightarrow A[\boldsymbol{x}]/(\boldsymbol{f}) \longrightarrow A[\boldsymbol{x}]/(z,\boldsymbol{f}) \longrightarrow 0,$$

and note that $zA[\boldsymbol{x}]/(\boldsymbol{f})$ is nonzero, and that it remains nonzero upon inverting any nonzero element of $A$. In view of Lemma 4.14, after replacing $A$ by its localization at one element, we may assume that the above is a split exact sequence of free $A$-modules.

Let $\mathfrak{m}$ be a maximal ideal of $A$. The field $\kappa := A/\mathfrak{m}$ is a finitely generated $\mathbb{Z}$-algebra, hence a finite field. Equation (4.15.1) implies that the image of $z$ is in the integral closure of the image of $\overline{(\boldsymbol{f})^n}$ in the polynomial ring $\kappa[\boldsymbol{x}]$, though not in the image of $(\boldsymbol{f})$, since

$$zA[\boldsymbol{x}]/(\boldsymbol{f}) \otimes_A \kappa$$

must be nonzero by the freeness hypotheses. This contradicts the Briançon-Skoda theorem for regular rings of positive characteristic, Theorem 4.9. ☐

*Proof of Theorem 4.15 in the affine case.* When the regular ring $R$ is finitely generated over a field $\mathbb{F}$, write it as a homomorphic image of a polynomial ring $T := \mathbb{F}[x_1,\dots,x_d]$, say

$$R \;=\; T/(g_1,\dots,g_m).$$

Let $\boldsymbol{f} := f_1,\dots,f_n$ be elements of $T$ that map to generators of the ideal $\mathfrak{a} \subseteq R$. If the statement of the theorem is false, then there exists $z \in T$ whose image in $R$ belongs to the set $\overline{\mathfrak{a}^n} \smallsetminus \mathfrak{a}$. Hence there exist elements $a_k \in (\boldsymbol{f})^{nk}$ and $h_1,\dots,h_m \in (g_1,\dots,g_m)$ such that

$$z^l + a_1 z^{l-1} + \cdots + a_l \;=\; h_1 g_1 + \cdots + h_m g_m. \tag{4.15.2}$$

Let $A$ be a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{F}$ containing the coefficients of $z, f_i$, $g_i$, $a_i$, and $h_i$ as polynomials in $x_1,\dots,x_d$, and also the coefficients of polynomials needed to express $a_k$ as an element of $(\boldsymbol{f})^{nk}$ for each $k$. Consider the $A$-algebra

$$R_A := A[x_1,\dots,x_d]/(g_1,\dots,g_m).$$

Let $\operatorname{frac} A$ denote the fraction field of $A$. The inclusion $\operatorname{frac} A \longrightarrow \mathbb{F}$ is a flat homomorphism of $A$-modules so, upon tensoring with $R_A$, we have a flat homomorphism

$$R_A \otimes_A \operatorname{frac} A \longrightarrow R_A \otimes_A \mathbb{F} \cong R.$$

The ring $R$ is regular, so $R_A \otimes_A \operatorname{frac} A$ is regular as well. Since $\operatorname{frac} A$ is a field of characteristic zero, it follows that

$$\operatorname{frac} A \longrightarrow R_A \otimes_A \operatorname{frac} A$$

is geometrically regular; since $R_A \otimes_A \operatorname{frac} A$ is finitely presented over $\operatorname{frac} A$, this map is smooth. After inverting an element of $A$, we may assume that $A \longrightarrow R_A$ is smooth.

After enlarging $A$ by inverting finitely many elements, we may assume that each of

$$0 \longrightarrow \boldsymbol{f}R_A \longrightarrow R_A \longrightarrow R_A/\boldsymbol{f}R_A \longrightarrow 0,$$

$$0 \longrightarrow (z,\boldsymbol{f})R_A \longrightarrow R_A \longrightarrow R_A/(z,\boldsymbol{f})R_A \longrightarrow 0,$$

$$0 \longrightarrow \boldsymbol{f}R_A \longrightarrow (z,\boldsymbol{f})R_A \longrightarrow (z,\boldsymbol{f})R_A/\boldsymbol{f}R_A \longrightarrow 0,$$

is a split exact sequence of free $A$-modules. Note that since the image of $z$ in $R$ does not belong to the ideal $\boldsymbol{f}R$, the module $(z,\boldsymbol{f})R_A/\boldsymbol{f}R_A$ must be nonzero. Let $\mathfrak{m}$ be a maximal ideal of $A$, in which case $\kappa := A/\mathfrak{m}$ is a finite field. Since $A \longrightarrow R_A$ is smooth, and smoothness is preserved under base change, we see that $R_\kappa := R \otimes_A \kappa$ is regular. The freeness hypotheses ensure that $\boldsymbol{f}R_A \otimes_A \kappa$ and $(z,\boldsymbol{f})R_A \otimes_A \kappa$ are ideals of $R_\kappa$, and also give the isomorphism

$$\frac{(z,\boldsymbol{f})R_A}{\boldsymbol{f}R_A} \otimes_A \kappa \cong \frac{(z,\boldsymbol{f})R_\kappa}{\boldsymbol{f}R_\kappa}.$$

Moreover, by freeness, this module is nonzero, i.e., the image of $z$ is not in the ideal $\boldsymbol{f}R_\kappa$. However, (4.15.2) ensures that the image of $z$ is in the integral closure of the ideal

$$(\boldsymbol{f})^n R_\kappa.$$

This contradicts the positive characteristic statement, Theorem 4.9. $\square$

For the general case of regular rings containing a field of characteristic zero, we invoke a rather deep result, known as *General Néron Desingularization* or *Néron-Popescu Desingularization* [Po, SwR]:

**Theorem 4.16.** *Let $\varphi\colon R \longrightarrow S$ be a geometrically regular homomorphism of Noetherian rings. Then $S$ is a direct limit of smooth $R$-algebras.*

*Hence, if $\varphi$ factors through a finitely generated $R$-algebra $R'$, then there exists an $R'$-algebra $T$ such that $R \longrightarrow T$ is smooth, and $\varphi$ factors as*

$$R \longrightarrow R' \longrightarrow T \longrightarrow S.$$

The result was first proved by Néron [Né] in the case that $R$ and $S$ are discrete valuation rings, hence the name. The case of interest for us is where $R$ is the localization of a polynomial ring over a field, and $\varphi\colon R \longrightarrow \widehat{R}$ the map to its completion. The map $\varphi$ is geometrically regular as $R$ is excellent, and General Néron Desingularization applies. This special case, and the corresponding statement for a polynomial ring over a discrete valuation ring, were also proved by Artin and Rotthaus [ArR]:

**Theorem 4.17.** *Let $R = \mathbb{F}[x_1,\dots,x_d]_{\mathfrak{m}}$, i.e., $R$ is the localization of a polynomial ring at its homogeneous maximal ideal $\mathfrak{m}$. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of $R$. Then, given a finitely generated $R$-subalgebra $R'$ of $\widehat{R}$, the inclusion $R \longrightarrow \widehat{R}$ factors as*

$$R \longrightarrow R' \longrightarrow T \longrightarrow \widehat{R},$$

*where $R \longrightarrow T$ is smooth.*

We mention that $T \longrightarrow \widehat{R}$ may not be injective, and that $T$ may have dimension greater than $R$. The crucial part is that $T$ is a finitely generated $R$-algebra that is regular.

*Proof of Theorem 4.15 completed.* Suppose the assertion is false; then there exists

$$z \in \overline{\mathfrak{a}^n} \smallsetminus \mathfrak{a}.$$

Let $\mathfrak{p}$ be a prime ideal containing $(\mathfrak{a} : z)$. Localizing at $\mathfrak{p}$ and completing, we may assume that the counterexample is in a power series ring $\mathbb{F}[[x_1,\dots,x_d]]$, where $\mathbb{F}$ is a field of characteristic zero. Set $R := \mathbb{F}[x_1,\dots,x_d]_{(x_1,\dots,x_d)}$.

Let $R'$ be a finitely generated $R$-subalgebra of $\widehat{R} = \mathbb{F}[[x_1,\dots,x_d]]$ that contains the element $z$, a fixed generating set for the ideal $\mathfrak{a}$, and elements of $\widehat{R}$ occurring in an equation demonstrating that $z \in \overline{\mathfrak{a}^n}$ in terms of the chosen generating set for $\mathfrak{a}$.

By Theorem 4.17, the maps $R \longrightarrow \widehat{R}$ factors as

$$R \longrightarrow R' \longrightarrow T \longrightarrow \widehat{R},$$

where $R \longrightarrow T$ is smooth. In particular, the ring $T$ is regular, and the images of $z$ and $\mathfrak{a}$ in $T$ also yield a counterexample, say

$$z_0 \in \overline{\mathfrak{a}_0^n} \smallsetminus \mathfrak{a}_0.$$

Note that $T$ is a regular ring of the form $W^{-1}B$, where $B$ is a finitely generated algebra over the field $\mathbb{F}$. Since $W^{-1}B$ is regular, the multiplicative set $W$ contains an element $a$ of the defining ideal of the singular locus of $B$. The element $z_0$, a generating set for $\mathfrak{a}_0$, and the elements occurring in one equation implying that $z_0 \in \overline{\mathfrak{a}_0^n}$ involve finitely many elements, hence finitely many denominators from $W$. Take $b$ to be the product of these denominators. We then obtain a counterexample in the ring $B_{ab}$, which is a regular ring finitely generated over the field $\mathbb{F}$. But we have already proved the Briançon-Skoda theorem in this case. $\square$

It should be mentioned that this approach is not the only way to deduce characteristic zero results from positive characteristic theorems: Schoutens [ScH] uses model-theoretic methods to obtain the Briançon-Skoda theorem for power series rings $\mathbb{C}[[X_1,\dots,X_d]]$ using the characteristic $p$ version, Theorem 4.9.

We conclude with an example, due to Hochster:

**Example 4.18.** Let $f,g,h$ be elements of a polynomial ring $\mathbb{F}[x,y]$. Then the Briançon-Skoda theorem implies that

$$f^2g^2h^2 \in (f^3,g^3,h^3)$$

as follows: After enlarging the field $\mathbb{F}$, the ideal $(f^3,g^3,h^3)$ has a *reduction* $\mathfrak{a}$ generated by two elements, i.e., for each $k$, the ideals $(f^3,g^3,h^3)^k$ and $\mathfrak{a}^k$ have the same integral closure. The Briançon-Skoda theorem, applied to $\mathfrak{a}$, gives

$$\overline{\mathfrak{a}^2} \subseteq \mathfrak{a},$$

leaving one with the easy verification that

$$f^2g^2h^2 \in \overline{(f^3,g^3,h^3)^2}. \qquad \blacksquare$$

## 5. Symbolic powers

Let $\mathfrak{p}$ be a prime ideal in a ring $R$. The *symbolic powers* of $\mathfrak{p}$ are defined as

$$\mathfrak{p}^{(n)} := \mathfrak{p}^n R_{\mathfrak{p}} \cap R,$$

for $n \geqslant 1$. When $R$ is Noetherian, the symbolic power $\mathfrak{p}^{(n)}$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^n$. The definition above immediately translates as

$$\mathfrak{p}^{(n)} \;=\; \{r \in R \mid sr \in \mathfrak{p}^n \text{ for some } s \notin \mathfrak{p}\}.$$

To motivate the definition, consider $f(x) = x^3(x-1)^2$ in $\mathbb{C}[x]$. One would readily agree that this polynomial vanishes to order 3 at the point $0 \in \mathbb{C}$, but not to order 4. The point 0 is the variety defined by $\mathfrak{m} := (x)$ and, indeed, the order of vanishing is measured by the fact that $f \in \mathfrak{m}^3$ while $f \notin \mathfrak{m}^4$. Symbolic powers provide the right extension of this:

Let $\mathfrak{p}$ be a prime ideal in the ring $R := \mathbb{C}[x_1,\dots,x_d]$. Then $\mathfrak{p}^{(n)}$ is the set of polynomials in $R$ that vanish to order at least $n$ along the variety defined by $\mathfrak{p}$ in $\mathbb{C}^d$, i.e.,

$$\mathfrak{p}^{(n)} \;=\; \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m}^n,$$

where the intersection is over the maximal ideals of $R$ that contain $\mathfrak{p}$. This is due to Zariski and Nagata, and holds more generally in a polynomial ring over an algebraically closed field, see [EH] or [Ei, Theorem 3.14].

**Example 5.1.** Take $V$ to be the variety of $3 \times 3$ complex matrices of rank at most 1. Setting $R := \mathbb{C}[X]$, where $X$ is a $3 \times 3$ matrix of indeterminates, $V$ is defined by the ideal $\mathfrak{p}$ generated by the size 2 minors of $X$. The ideal $\mathfrak{p}$ is indeed prime, and contains the determinant $\det X$. Since $\det X$ is a polynomial of degree 3, and $\mathfrak{p}$ is generated by quadrics,

$$\det X \;\notin\; \mathfrak{p}^2.$$

We claim that, however, $\det X \in \mathfrak{p}^{(2)}$.

Set $[a_1\ \dots\ a_r \mid b_1\ \dots\ b_r]$ to be the determinant of the submatrix with rows $a_1,\dots,a_r$ and columns $b_1,\dots,b_r$. The following identity is an example of a *straightening law*:

$$[1\ 2 \mid 1\ 3]\cdot[1\ 3 \mid 1\ 2] \;=\; [1\ 2 \mid 1\ 2]\cdot[1\ 3 \mid 1\ 3] - [1 \mid 1]\cdot[1\ 2\ 3 \mid 1\ 2\ 3].$$

Since $[1 \mid 1] = x_{11}$ is not an element of the ideal $\mathfrak{p}$, it follows that $\det X = [1\ 2\ 3 \mid 1\ 2\ 3]$ belongs to $\mathfrak{p}^{(2)}$ as claimed. Thus, $\det X$ vanishes to order at least 2 on each matrix of rank at most 1. For a different take, see [Ei, § 3.9.1] ■

Our main focus will be symbolic powers in regular rings, but first:

**Exercise 5.2.** Prove that the symbolic powers of a maximal ideal are its ordinary powers.

**Exercise 5.3.** Construct prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ in $\mathbb{F}[w,x,y,z]/(wx-yz)$ such that $\mathfrak{p}^{(2)} \nsubseteq \mathfrak{q}^{(2)}$, better still, with $\mathfrak{p}^{(n)} \nsubseteq \mathfrak{q}^{(n)}$ for each $n \geqslant 2$.

It turns out that if $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals in a regular ring, then $\mathfrak{p}^{(n)} \subseteq \mathfrak{q}^{(n)}$ for all $n \geqslant 1$. Exercise: Why should you expect this in $\mathbb{C}[x_1,\dots,x_d]$?

**Exercise 5.4.** Determine the symbolic powers of the ideal $(x)$ in $\mathbb{F}[x,\ y]/(xy)$, and of the ideal $(x^2,\ xy)$ in $\mathbb{F}[x^2,\ xy,\ y^2]$.

**Exercise 5.5.** For a prime $\mathfrak{p}$ of positive height in a Noetherian ring, prove that $\mathfrak{p}^m \subseteq \mathfrak{p}^{(n)}$ if and only if $n \leqslant m$.

We next analyze containments of the form $\mathfrak{p}^{(m)} \subseteq \mathfrak{p}^n$, which turns out to be much more subtle. The following remarkable result was proved by Ein, Lazarsfeld, and Smith [ELS] for regular rings essentially of finite type over $\mathbb{C}$; it was subsequently extended by Hochster and Huneke [HH8] to regular rings containing a field. A recent preprint of Ma and Schwede uses perfectoid methods to settle the case of mixed characteristic as well, [MaS].

**Theorem 5.6.** *Let $\mathfrak{p}$ be a prime ideal of height $h$ in a regular ring that contains a field. Then, for each $n \geqslant 1$, one has*

$$\mathfrak{p}^{(hn)} \ \subseteq \ \mathfrak{p}^n.$$

Swanson [Sw3] had previously proved—for a larger class of rings—the existence of a constant $c$, depending on the prime $\mathfrak{p}$, such that $\mathfrak{p}^{(cn)} \subseteq \mathfrak{p}^n$ for each $n \geqslant 1$. The above theorem says that in equicharacteristic regular rings, one may take $c$ to be the height of $\mathfrak{p}$.

*Proof.* We stick to the positive characteristic case here. A regular ring is a product of domains; working with individual factors, assume that the regular ring $R$ is a domain of characteristic $p > 0$. Assume $\mathfrak{p} \neq 0$, as there is nothing to say otherwise. We claim that for each $q = p^e$, one has

$$\mathfrak{p}^{(hq)} \ \subseteq \ \mathfrak{p}^{[q]}.$$

It suffices to verify that the displayed containment holds after localizing at each associated prime of $\mathfrak{p}^{[q]}$. By the flatness of the Frobenius endomorphism of $R$, it follows that $\mathfrak{p}$ is the unique associated prime of $\mathfrak{p}^{[q]}$, see Lemma 5.7, so the claim follows once we verify that

$$\mathfrak{p}^{hq} R_{\mathfrak{p}} \ = \ \mathfrak{p}^{(hq)} R_{\mathfrak{p}} \ \subseteq \ \mathfrak{p}^{[q]} R_{\mathfrak{p}}.$$

But $R_{\mathfrak{p}}$ is a regular local ring of dimension $h$, so its maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ is generated by $h$ elements, and the above containment holds by the pigeonhole principle.

Now suppose $u \in \mathfrak{p}^{(hn)}$ for a fixed $n$. Let $q = p^e$ be arbitrary, and write $q = an + r$ for integers $a \geqslant 0$ and $0 \leqslant r \leqslant n-1$. Then $u^a \in \mathfrak{p}^{(han)}$, so

$$u^a \mathfrak{p}^{h(n-1)} \ \subseteq \ \mathfrak{p}^{(h(an+n-1))} \ \subseteq \ \mathfrak{p}^{(h(an+r))} \ = \ \mathfrak{p}^{(hq)} \ \subseteq \ \mathfrak{p}^{[q]}.$$

Taking $n$-th powers, one has

$$\mathfrak{p}^{h(n-1)n} u^{an} \ \subseteq \ \left(\mathfrak{p}^{[q]}\right)^n \ = \ \left(\mathfrak{p}^n\right)^{[q]}.$$

Since $q \geqslant an$, it follows that

$$\mathfrak{p}^{h(n-1)n} u^q \ \subseteq \ \left(\mathfrak{p}^n\right)^{[q]}.$$

Taking $c \neq 0$ in $\mathfrak{p}^{h(n-1)n}$, the above display implies that $cu^q \in \left(\mathfrak{p}^n\right)^{[q]}$ for all $q = p^e$. Hence

$$u \ \in \ \left(\mathfrak{p}^n\right)^* \ = \ \mathfrak{p}^n. \qquad \square$$

**Lemma 5.7.** *Let $\mathfrak{p}$ be a prime ideal in a regular ring $R$ of prime characteristic $p > 0$. Then, for each $q = p^e$, one has*

$$\operatorname{Ass} R/\mathfrak{p}^{[q]} \;=\; \{\mathfrak{p}\}.$$

*Proof.* An element $r \in R \smallsetminus \mathfrak{p}$ acts injectively on $R/\mathfrak{p}$, i.e.,

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{\;r\;} R/\mathfrak{p}$$

is exact. But then, applying the exact functor $F^e(R) \otimes_R -$, so is

$$0 \longrightarrow R/\mathfrak{p}^{[q]} \xrightarrow{\;r^q\;} R/\mathfrak{p}^{[q]}.$$

Since $r^q$ acts injectively on $R/\mathfrak{p}^{[q]}$, so does $r$. □

**Exercise 5.8.** Let $R$ be a regular ring of prime characteristic $p > 0$, and $\mathfrak{a}$ an ideal of $R$. Prove that for each $q = p^e$, one has

$$\operatorname{Ass} R/\mathfrak{a}^{[q]} \;=\; \operatorname{Ass} R/\mathfrak{a}.$$

**Definition 5.9.** Let $\mathfrak{a}$ be an ideal in a Noetherian ring $R$. The $n$-th *symbolic power* of $\mathfrak{a}$ is

$$\mathfrak{a}^{(n)} := \bigcap_{\mathfrak{p} \in \operatorname{Ass} R/\mathfrak{a}} \mathfrak{a}^n R_{\mathfrak{p}} \cap R.$$

**Exercise 5.10.** Verify that $\mathfrak{a}^{(n)}$ equals $\big(\mathfrak{a}^n W^{-1}R\big) \cap R$, where $W$ is the complement of the union of the associated primes of $\mathfrak{a}$, and that $\mathfrak{a}^{(1)}$ equals $\mathfrak{a}$.

We record an extension of Theorem 5.6 due to Hochster and Huneke [HH8]:

**Theorem 5.11.** *Let $\mathfrak{a}$ be a radical ideal in a regular ring that contains a field. Set $h$ to be the largest height of a minimal prime of $\mathfrak{a}$. Then, for each $n \geqslant 1$ and $k \geqslant 0$, one has*

$$\mathfrak{a}^{(hn+kn)} \;\subseteq\; \big(\mathfrak{a}^{(k+1)}\big)^n.$$

*Proof.* We discuss the positive characteristic case here; see Remark 5.13 for characteristic zero. Proceeding as in the proof of Theorem 5.6, take the regular ring $R$ to be a domain of characteristic $p > 0$. We claim that for each $q = p^e$, one has

$$\mathfrak{a}^{(hq+kq)} \;\subseteq\; \big(\mathfrak{a}^{(k+1)}\big)^{[q]}. \tag{5.11.1}$$

An associated primes of $\mathfrak{a}^{(k+1)}$ must be an associated prime of $\mathfrak{a}$, and hence a minimal prime of $\mathfrak{a}$, since $\mathfrak{a}$ is radical. By Exercise 5.8, the associated primes of $\big(\mathfrak{a}^{(k+1)}\big)^{[q]}$ are the same as those of $\mathfrak{a}^{(k+1)}$. Let $\mathfrak{p}$ be one of these associated primes; it suffices to verify that

$$\mathfrak{a}^{(hq+kq)} R_{\mathfrak{p}} \;\subseteq\; \big(\mathfrak{a}^{(k+1)}\big)^{[q]} R_{\mathfrak{p}}.$$

Let $W$ denote the complement of the union of the associated primes of $\mathfrak{a}$. Since $R_{\mathfrak{p}}$ is a localization of $W^{-1}R$, it suffices to verify that

$$\mathfrak{a}^{hq+kq} R_{\mathfrak{p}} \;\subseteq\; \big(\mathfrak{a}^{k+1}\big)^{[q]} R_{\mathfrak{p}},$$

which is Exercise 5.12, and completes the proof of (5.11.1).

Let $u \in \mathfrak{a}^{(hn+kn)}$ for some $n,k$. Write $q = an+r$, where $a \geqslant 0$ and $0 \leqslant r \leqslant n-1$. Then

$$u^a \mathfrak{a}^{(h+k)(n-1)} \ \subseteq\ \mathfrak{a}^{((h+k)(an+n-1))} \ \subseteq\ \mathfrak{a}^{((h+k)(an+r))} \ =\ \mathfrak{a}^{(hq+kq)} \ \subseteq\ \left(\mathfrak{a}^{(k+1)}\right)^{[q]}.$$

Taking $n$-th powers, one has

$$\mathfrak{a}^{(h+k)(n-1)n} u^{an} \ \subseteq\ \left(\left(\mathfrak{a}^{(k+1)}\right)^n\right)^{[q]}.$$

It follows that

$$\mathfrak{a}^{(h+k)(n-1)n} u^{q} \ \subseteq\ \left(\left(\mathfrak{a}^{(k+1)}\right)^n\right)^{[q]},$$

and hence that $u \in \left(\mathfrak{a}^{(k+1)}\right)^n$. $\square$

**Exercise 5.12.** Suppose $\mathfrak{a}$ is an ideal generated by $h$ elements in a ring of prime characteristic $p > 0$. Verify that for each $k \geqslant 0$ and $q = p^e$, one has

$$\mathfrak{a}^{hq+kq} \ \subseteq\ \left(\mathfrak{a}^{[q]}\right)^{k+1} \ =\ \left(\mathfrak{a}^{k+1}\right)^{[q]}.$$

**Remark 5.13.** We refer to [HH8, § 4] for the characteristic zero case; the broad steps are as in the proof of the Briançon-Skoda theorem: (i) If there is a counterexample, localize and complete so as to have a counterexample in a power series ring $\widehat{R} = \mathbb{F}[[x_1, \dots, x_d]]$, where $\mathbb{F}$ is a field of characteristic zero. (ii) Use General Néron Desingularization to descend to a counterexample in a regular finitely generated $\mathbb{F}$-algebra. (iii) Collect the relevant data in a finitely generated $\mathbb{Z}$-algebra, and reduce to the case of positive characteristic. That being said, one certainly needs to be careful: for a start, since we have not assumed that $R$ is excellent, the ideal $\mathfrak{a}\widehat{R}$ need not be radical. One way around this is drop the hypothesis that $\mathfrak{a}$ is radical, and go for greater generality as in [HH8, Theorem 4.4]:

**Theorem 5.14.** *Let $\mathfrak{a}$ be an ideal in a regular ring that contains a field. Set $h$ to be the largest analytic spread of $\mathfrak{a}R_\mathfrak{p}$, as $\mathfrak{p}$ runs through the associated primes of $\mathfrak{a}$. Then, for each $n \geqslant 1$ and $k \geqslant 0$, one has*

$$\mathfrak{a}^{(hn+kn)} \ \subseteq\ \left(\mathfrak{a}^{(k+1)}\right)^n.$$

Such extensions notwithstanding, there is likely room for improvement:

**Example 5.15.** Take the polynomial ring $\mathbb{F}[x,y,z]$, where $\mathbb{F}$ is a field, and set

$$\mathfrak{a} := (x,y) \cap (y,z) \cap (z,x).$$

Then each minimal prime of $\mathfrak{a}$ has height 2, so the theorem gives $\mathfrak{a}^{(2n)} \subseteq \mathfrak{a}^n$ for each $n$, in particular, $\mathfrak{a}^{(4)} \subseteq \mathfrak{a}^2$. However, it is readily seen that

$$\mathfrak{a}^{(3)} \ =\ (x,y)^3 \cap (y,z)^3 \cap (z,x)^3 \ =\ (x^3y^3,\ y^3z^3,\ z^3x^3,\ xy^2z^2,\ x^2yz^2,\ x^2y^2z),$$

so that $\mathfrak{a}^{(3)} \subseteq \mathfrak{a}^2$. ■

**Example 5.16.** Take $\mathfrak{p}$ to be the kernel of the $\mathbb{F}$-algebra surjection $\mathbb{F}[x,y,z] \longrightarrow \mathbb{F}[t^3,t^4,t^5]$ that sends the indeterminates $x$, $y$, $z$ to $t^3$, $t^4$, $t^5$ respectively. Then $\mathfrak{p}$ is generated by the size 2 minors of the matrix

$$\begin{pmatrix} x & y & z \\ y & z & x^2 \end{pmatrix},$$

as may be confirmed by *Macaulay2*, [GS], which also shows that

$$\mathfrak{p}^{(2)} \;=\; \mathfrak{p}^2 + (x^5 + xy^3 + z^3 - 3x^2yz),$$

so that $\mathfrak{p}^{(2)} \not\subseteq \mathfrak{p}^2$. It turns out that $\mathfrak{p}^{(3)} \subseteq \mathfrak{p}^2$. ■

**Question 5.17** (Huneke)**.** For a prime ideal $\mathfrak{p}$ of height 2 in an equicharacteristic regular ring, does the inclusion $\mathfrak{p}^{(3)} \subseteq \mathfrak{p}^2$ always hold?

A more general statement was conjectured subsequently in the graded setting, [BDH$^+$, Conjecture 8.4.3]:

**Conjecture 5.18** (Harbourne)**.** Let $\mathfrak{a}$ be a homogeneous radical ideal in a polynomial ring over a field. If $\mathfrak{a}$ has height $h$, then for each integer $n \geqslant 1$ one has

$$\mathfrak{a}^{(hn-h+1)} \;\subseteq\; \mathfrak{a}^n.$$

The conjecture holds for square-free monomial ideals [BDH$^+$, Example 8.4.5], for ideals defining general points in $\mathbb{P}^2$ [BoH], and for ideals defining general points in $\mathbb{P}^3$ [Dum]. Note that if $h = 2 = n$, the conjecture says that $\mathfrak{a}^{(3)} \subseteq \mathfrak{a}^2$. This has been verified for ideals defining space monomial curves as in Example 5.16, see [Gri]. However, it is false in general, as demonstrated by Dumnicki, Szemberg, and Tutaj-Gasińska [DST]:

**Example 5.19.** Consider the height 2 ideal

$$\mathfrak{a} := \big(x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)\big)$$

in the polynomial ring $\mathbb{C}[x,y,z]$. It is proved in [DST] that

$$(x^3-y^3)(y^3-z^3)(z^3-x^3) \;\in\; \mathfrak{a}^{(3)} \smallsetminus \mathfrak{a}^2.$$

Note that the ideal is homogeneous; it defines a configuration of 12 points in $\mathbb{P}^2_{\mathbb{C}}$, lying at the pairwise intersections of 9 lines with the property that each point lies on 3 lines, and each line passes through 4 points. Briefly, the 12 points are not general! ■

Subsequently, Harbourne and Seceleanu [HS] showed that for any $k \geqslant 3$, and $\mathbb{F}$ a field of characteristic other than 2 that contains $k$ distinct $k$-th roots of unity, the ideal

$$\mathfrak{a} := \big(x(y^k-z^k),\ y(z^k-x^k),\ z(x^k-y^k)\big)$$

in $\mathbb{F}[x,y,z]$, has the property that $\mathfrak{a}^{(3)} \not\subseteq \mathfrak{a}^2$. We are not aware of any counterexamples to Conjecture 5.18 where the ideal $\mathfrak{a}$ is prime, or where $\mathfrak{a}$ is a radical ideal of height 3. Closer to the theme of these lectures, one has the recent results of Grifo and Huneke [GH]; they prove, for example, that Conjecture 5.18 holds for ideals defining $F$-pure rings:

**Theorem 5.20.** *Let $R$ be a regular ring of positive characteristic, and $\mathfrak{a}$ an ideal such that $R/\mathfrak{a}$ is $F$-pure. Set $h$ to be the largest height of a minimal prime of $\mathfrak{a}$. Then, for each $n \geqslant 1$, one has*

$$\mathfrak{a}^{(hn-h+1)} \subseteq \mathfrak{a}^n.$$

*Proof.* The theorem reduces to the local case; let $(R,\mathfrak{m})$ be a regular local ring of characteristic $p>0$. Suppose the assertion is false for a fixed $n$, then

$$\mathfrak{a}^n : \mathfrak{a}^{(hn-h+1)} \subseteq \mathfrak{m},$$

and so, for each $q=p^e$, one has

$$\left(\mathfrak{a}^n : \mathfrak{a}^{(hn-h+1)}\right)^{[q]} \subseteq \mathfrak{m}^{[q]}.$$

We claim that

$$\mathfrak{a}^{[q]} : \mathfrak{a} \subseteq \left(\mathfrak{a}^n : \mathfrak{a}^{(hn-h+1)}\right)^{[q]} \tag{5.20.1}$$

for $q \gg 0$. Assuming the claim, the previous two containments then combine to give

$$\mathfrak{a}^{[q]} : \mathfrak{a} \subseteq \mathfrak{m}^{[q]},$$

for $q \gg 0$, which contradicts the $F$-purity of $R/\mathfrak{a}$ in view of Theorem 3.32.

To prove (5.20.1), pick $x$ in $\mathfrak{a}^{[q]} : \mathfrak{a}$. In view of Exercise 2.9, we need to show that

$$x\left(\mathfrak{a}^{(hn-h+1)}\right)^{[q]} \subseteq (\mathfrak{a}^n)^{[q]} = \left(\mathfrak{a}^{[q]}\right)^n$$

for $q \gg 0$. But

$$x\left(\mathfrak{a}^{(hn-h+1)}\right)^{[q]} \subseteq x\mathfrak{a}\left(\mathfrak{a}^{(hn-h+1)}\right)^{q-1} \subseteq \mathfrak{a}^{[q]}\left(\mathfrak{a}^{(hn-h+1)}\right)^{q-1},$$

so it suffices to show that

$$\left(\mathfrak{a}^{(hn-h+1)}\right)^{q-1} \subseteq \left(\mathfrak{a}^{[q]}\right)^{n-1}$$

for $q \gg 0$. Theorem 5.11, with appropriate choices of $k$ and $n$, gives

$$\mathfrak{a}^{((hq+h-1)(n-1))} \subseteq \left(\mathfrak{a}^{(hq)}\right)^{n-1}.$$

On the other hand, (5.11.1) gives $\mathfrak{a}^{(hq)} \subseteq \mathfrak{a}^{[q]}$, and combining the two, one has

$$\mathfrak{a}^{((hq+h-1)(n-1))} \subseteq \left(\mathfrak{a}^{[q]}\right)^{n-1}.$$

Thus, it suffices to verify that for $q \gg 0$ one has

$$\left(\mathfrak{a}^{(hn-h+1)}\right)^{q-1} \subseteq \mathfrak{a}^{((hq+h-1)(n-1))},$$

for which it is enough to check that

$$(hn-h+1)(q-1) \geqslant (hq+h-1)(n-1),$$

and this is indeed the case for $q \gg 0$. $\square$

## 6. Appendix

**Complete local rings.** Every ring $R$ admits a canonical ring homomorphism $\varphi\colon \mathbb{Z} \longrightarrow R$, where $\varphi(1)$ is the unit element of $R$. The kernel of $\varphi$ is an ideal of $\mathbb{Z}$ generated be a nonnegative integer that is the *characteristic* of $R$, denoted here as $\operatorname{char} R$.

A local ring $(R,\mathfrak{m})$ is *equicharacteristic* if the characteristic of $R$ equals that of the field $\mathbb{F} := R/\mathfrak{m}$. For a local ring $(R,\mathfrak{m},\mathbb{F})$ the possible values of $\operatorname{char} R$ and $\operatorname{char}\mathbb{F}$ are:

(1) $\operatorname{char} R = 0 = \operatorname{char}\mathbb{F}$, in which case $\mathbb{Q} \subseteq R$,
(2) $\operatorname{char} R = p = \operatorname{char}\mathbb{F}$, for $p$ a positive prime integer, in which case $\mathbb{Z}/p \subseteq R$,
(3) $\operatorname{char} R = 0$ and $\operatorname{char}\mathbb{F} = p > 0$, for example $R = \mathbb{Z}_{(p)}$,
(4) $\operatorname{char} R = p^e > 0$ and $\operatorname{char}\mathbb{F} = p$ where $e \geqslant 2$, in which case $R$ is not reduced.

Suppose $(R,\mathfrak{m})$ is a local ring containing a field. A subring $\mathbb{F}$ of $R$, that is a field, is a *coefficient field* if the composition

$$\mathbb{F} \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$$

is an isomorphism. The following structure theorems are due to Cohen [Co]:

**Theorem 6.1.** *Let $(R,\mathfrak{m})$ be a complete local ring containing a field. Then $R$ contains a coefficient field $\mathbb{F}$. Moreover:*

(1) *The ring $R$ is a homomorphic image of a formal power series ring over $\mathbb{F}$.*
(2) *If $x_1,\dots,x_d$ is a system of parameters for $R$, then the subring $A := \mathbb{F}[[x_1,\dots,x_d]]$ is isomorphic to a formal power series ring, and $R$ is a finitely generated $A$-module.*
(3) *$R$ is regular if and only if it isomorphic to a formal power series ring over $\mathbb{F}$.*

In the case where $(R,\mathfrak{m})$ does not contain a field, for the sake of simplicity, we shall work in the setting where $R$ is an integral domain. This ensures that the only possibility is $\operatorname{char} R = 0$ and $\operatorname{char} R/\mathfrak{m} = p > 0$; the ring $R$ is said to be of *mixed characteristic $p$*. The role of a coefficient field is now played by that of a discrete valuation ring $(V, pV)$, that serves as a *coefficient ring*. A regular local ring $(R,\mathfrak{m})$ of mixed characteristic $p$ is *unramified* if $p \notin \mathfrak{m}^2$, and it is *ramified* if $p \in \mathfrak{m}^2$.

**Theorem 6.2.** *Let $(R,\mathfrak{m})$ be a complete local domain of mixed characteristic $p$. Then there exists a discrete valuation ring $(V, pV)$ that is a subring of $R$, such $V \subseteq R$ induces an isomorphism $V/pV \longrightarrow R/\mathfrak{m}$. Moreover:*

(1) *The ring $R$ is a homomorphic image of a formal power series ring over $V$.*
(2) *If $p, x_2,\dots,x_d$ is a system of parameters for $R$, then the subring $A := V[[x_2,\dots,x_d]]$ is isomorphic to a formal power series ring over $V$, and $R$ is module-finite over $A$.*
(3) *The ring $R$ is an unramified regular local ring if and only if it is isomorphic to a formal power series ring over $V$.*
(4) *The ring $R$ is a ramified regular local ring if and only if it is isomorphic to*

$$V[[T_1,\dots,T_d]]/(p-f),$$

*where $f$ is an element in the square of the maximal ideal of $V[[T_1,\dots,T_d]]$.*

**Excellent rings.** The class of *excellent rings* was introduced by Grothendieck to circumvent some pathological behavior that can occur in the larger class of Noetherian rings. The precise definition is somewhat technical, but is satisfied by most Noetherian rings that are likely to be encountered in algebraic geometry, number theory, or several complex variables. For a detailed treatment and proofs of results summarized here, the reader may consult [Gro, § 7] or [Ma1]. The expository article [Ma2] provides a nice introduction to the theory of excellent rings of characteristic zero. Various examples of non-excellent rings due to Nagata are included as an appendix in his book [Na2].

**Definition 6.3.** A Noetherian ring $R$ is *excellent* if

(1) $R$ is universally catenary,
(2) for each $\mathfrak{p} \in \operatorname{Spec} R$, the formal fibers of $R_\mathfrak{p}$ are geometrically regular, and
(3) for every finitely generated $R$-algebra $S$, the regular locus of the ring $S$, i.e., the set

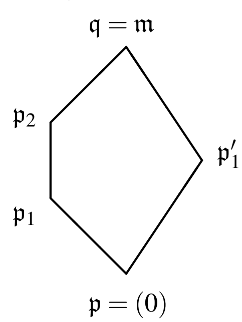
$$\{\mathfrak{p} \in \operatorname{Spec} S \mid S_\mathfrak{p} \text{ is a regular local ring}\},$$

is an open subset of $\operatorname{Spec} S$.

We now need to define some of the terms occurring above!

**Universally catenary rings.** A ring $R$ is *catenary* if for prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of $R$, all saturated chains of prime ideals joining $\mathfrak{p}$ and $\mathfrak{q}$ have the same length. A ring $R$ is *universally catenary* if every finitely generated $R$-algebra is catenary.

Attempts had been made to prove that all Noetherian rings were catenary, until Nagata constructed the first examples of noncatenary Noetherian rings in 1956. He constructed a local domain $(R, \mathfrak{m})$ of dimension 3 that is not catenary; $R$ has saturated chains of prime ideals joining $\mathfrak{p} = (0)$ and $\mathfrak{q} = \mathfrak{m}$ of lengths 2 and 3, as in the diagram below:




**Theorem 6.4** (Dimension formula)**.** *Let $R \subseteq S$ be domains such that $R$ is universally catenary and $S$ is a finitely generated $R$-algebra. Let $\mathfrak{q} \in \operatorname{Spec} S$ and set $\mathfrak{p} = \mathfrak{q} \cap R$. Then*

$$\operatorname{height} \mathfrak{q} + \operatorname{tr.deg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \;=\; \operatorname{height} \mathfrak{p} + \operatorname{tr.deg}_R S,$$

*where $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and $\kappa(\mathfrak{q}) = S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$, and* $\operatorname{tr.deg}_R S$ *denotes the transcendence degree of the fraction field of $S$ over the fraction field of $R$.*

Ratliff [Ra1, Ra2] showed that the dimension formula essentially characterizes universally catenary rings.

**Fibers and geometric regularity.** For a ring homomorphism $\varphi\colon R \longrightarrow S$, the *fiber* of $\varphi$ at a prime $\mathfrak{p}$ of $R$ is the ring

$$S \otimes_R \kappa(\mathfrak{p}),$$

which is an algebra over the field $\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Note that the inverse image of $\mathfrak{p}$ under the induced map $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is homeomorphic to $\operatorname{Spec} S \otimes_R \kappa(\mathfrak{p})$, which explains the use (or, at least, the misuse) of the term "fiber."

When $R$ is a domain, the *generic fiber* is the fiber over $(0)$ in $\operatorname{Spec} R$. For $(R, \mathfrak{m})$ local, the fiber over $\mathfrak{m}$ is the *closed fiber*, whereas the *formal fibers* of $R$ are the fibers of

$$R \longrightarrow \widehat{R},$$

where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of $R$. Since $\widehat{R/\mathfrak{a}} = \widehat{R}/\mathfrak{a}\widehat{R}$ for an ideal $\mathfrak{a}$ of $R$, the formal fibers of the ring $R/\mathfrak{a}$ are also formal fibers of $R$.

For $\mathbb{F}$ a field, a Noetherian $\mathbb{F}$-algebra $R$ is *geometrically regular* if $R \otimes_{\mathbb{F}} \mathbb{K}$ is a regular ring for each finite extension field $\mathbb{K}$ of $\mathbb{F}$, equivalently, for each finite purely inseparable extension field $\mathbb{K}$ of $\mathbb{F}$.

**Example 6.5.** Take $\mathbb{F} := \mathbb{F}_p(t)$ where $t$ is transcendental over $\mathbb{F}_p$. Then $\mathbb{F}[x]/(x^p - t)$ is a regular ring, but it is not geometrically regular over $\mathbb{F}$ since the ring

$$\frac{\mathbb{F}[x]}{(x^p - t)} \otimes_{\mathbb{F}} \mathbb{F}(t^{1/p}) \;\cong\; \frac{\mathbb{F}(t^{1/p})[x]}{(x-t)^p}$$

is not reduced. ■

On the other hand, if $R$ is a Noetherian $\mathbb{F}$-algebra, and $R \otimes_{\mathbb{F}} \mathbb{K}$ is a regular ring for some extension field $\mathbb{K}$ of $\mathbb{F}$, then $R$ must be regular; more generally, if $R \longrightarrow S$ is a faithfully flat homomorphism of Noetherian rings, and $S$ is regular, then $R$ must be regular.

A homomorphism $\varphi\colon R \longrightarrow S$ of Noetherian rings is *geometrically regular* if it is flat, and for each $\mathfrak{p} \in \operatorname{Spec} R$, the fiber

$$\kappa(\mathfrak{p}) \longrightarrow S \otimes_R \kappa(\mathfrak{p})$$

is geometrically regular. Note that this agrees with the notion of a geometrically regular algebra over a field defined above. The homomorphism $\varphi$ is *smooth* if it is geometrically regular, and $S$ is finitely presented over the image of $R$. Smoothness is preserved under base change: if $R \longrightarrow S$ is smooth, and $T$ an $R$-algebra, then

$$R \otimes_R T \longrightarrow S \otimes_R T$$

is also smooth.

The following explains why the Noetherian rings that we encounter are often excellent:

**Theorem 6.6.** *If a ring $R$ is obtained by adjoining finitely many variables to a field or to a complete discrete valuation ring, taking a homomorphic image, and localizing at some multiplicative set, then $R$ is an excellent ring. Moreover:*

(1) *Every complete local ring (in particular, every field) is excellent. The ring of convergent power series over $\mathbb{R}$ or $\mathbb{C}$ is excellent. A Dedekind domain whose field of fractions has characteristic zero (e.g., $\mathbb{Z}$) is excellent.*
(2) *A finitely generated algebra over an excellent ring is excellent; in particular, a homomorphic image of an excellent ring is excellent.*
(3) *A localization of an excellent ring is excellent.*

In characteristic $p > 0$, one also has the following theorem of Kunz [Ku1, Ku2]

**Theorem 6.7.** *Let $R$ be a Noetherian local ring of prime characteristic $p$. If the Frobenius endomorphism $F\colon R \longrightarrow R$ is finite, i.e., $R$ is module-finite over $R^p$, then $R$ is excellent.*

We record some properties of excellent rings:

**Theorem 6.8.** *If $R$ is excellent, then the normal locus and the Cohen-Macaulay locus*

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid R_\mathfrak{p} \text{ is normal}\} \quad \text{and} \quad \{\mathfrak{p} \in \operatorname{Spec} R \mid R_\mathfrak{p} \text{ is Cohen-Macaulay}\}$$

*are open subsets of* $\operatorname{Spec} R$.

Nagata defined a Noetherian ring $R$ to be *pseudo-geometric* if for each prime $\mathfrak{p}$ in $\operatorname{Spec} R$, and each finite extension field $\mathbb{F}$ of the field of fractions of $R/\mathfrak{p}$, the integral closure of $R/\mathfrak{p}$ in $\mathbb{F}$ is a finitely generated $R/\mathfrak{p}$-module. Examples of Noetherian rings that do not satisfy this property were first constructed by Akizuki [Ak]. In honor of the Japanese school of commutative algebra, Grothendieck renamed pseudo-geometric rings as *anneaux universellement japonais*, or universally Japanese rings [Gro, § 7.7]. At some point they were again renamed, and are now called *Nagata rings*.

**Theorem 6.9.** *An excellent ring is a Nagata ring.*

The excellence property also ensures that various properties of a local ring $R$ are inherited by its $\mathfrak{m}$-adic completion $\widehat{R}$, and this is the essence of the next theorem:

**Theorem 6.10.** *Let $(R, \mathfrak{m})$ be an excellent local ring, and $\widehat{R}$ its $\mathfrak{m}$-adic completion.*
(1) *If $R$ is reduced, so is $\widehat{R}$.*
(2) *If $R$ is a domain, then, by (1), $\widehat{R}$ is reduced ring. In this case, there is a bijection between the minimal primes of $\widehat{R}$ and the maximal ideals of $R'$, where $R'$ denotes the integral closure of $R$ in its field of fractions.*
*In particular, $\widehat{R}$ is a domain if and only if $R'$ is local.*
(3) *If $R$ is a normal ring, then so is $\widehat{R}$.*

**Exercise 6.11.** Set $R := \mathbb{Q}[x,y]_{(x,y)}/(x^2y^2 + x^6 + y^6)$. Verify that this is a domain, and determine the integral closure of $R$ in $\operatorname{frac} R$. Determine the minimal primes of $\widehat{R}$.

**Exercise 6.12.** Let $R$ be the localization of $\mathbb{R}[x]$ at the prime ideal $(x^2+1)$. Prove that $R$ does not have a coefficient field.

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