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Relative test elements for tight closure

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Abstract

Test ideals play a crucial role in the theory of tight closure developed by Melvin Hochster and Craig Huneke. Recently, Karen Smith showed that test ideals are closely related to certain multiplier ideals that arise in vanishing theorems in algebraic geometry. In this paper we develop a generalization of the notion of test ideals: for complete local rings R and S, where S is a module-finite extension of R, we define a *module of relative test elements* $T(S,R)$ which is a submodule of $\mathrm{Hom}_R(S,R)$.

MSC: 13A35; 13H10

1. Introduction

Throughout this paper, all rings are commutative, Noetherian, and contain a field of characteristic $p > 0$. The theory of *tight closure* was developed by Hochster and Huneke in [3]. Tight closure is a closure operation on ideals, i.e., for an ideal I of a ring R, the tight closure of I is a possibly larger ideal denoted by I^*. The test ideal is the ideal consisting of elements which, for every ideal I, multiply elements of I^* into the ideal I. A study of test ideals led to the uniform Artin–Rees theorems of Huneke [7] and their importance is also highlighted by the recent work [12] where it is established that for certain local rings of characteristic zero, the multiplier ideal (as defined in [9]) is a *universal test ideal.*

Let R and S be complete local rings where S is a module-finite extension of the subring R. We shall define the *module of relative test elements* $T(S,R)$ as a submodule of the module of R-linear homomorphisms $\mathrm{Hom}_R(S,R)$. This gives a generalization of

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test ideals in the following sense: for a complete local ring S, the module $T(S,S)$ is isomorphic to the test ideal of S. While we do not pursue it here the theory can also be developed when R and S are $\mathbb{N}$-graded rings over a perfect (or F-finite) field $R_0 = S_0 = K$.

If the subring R of S is Gorenstein, then $\mathrm{Hom}_R(S,R) \cong \omega_S$, the canonical module of S, and $T(S,R)$ may be viewed as a submodule of ω_S. We show that $T(S,R)$ does not depend on a specific choice of the Gorenstein subring R and is, in fact, the *parameter test module* defined in [11]. This is an F-submodule of ω_S in the terminology of Smith [11], and consequently has suitable localization properties.

2. Notation and definitions

Let R be a Noetherian ring of characteristic $p > 0$. The letter e denotes a variable nonnegative integer, and q its eth power, $q = p^e$. A reduced ring R, is said to be *F-finite* if $R^{1/p}$, the ring obtained by adjoining all pth roots of elements of R, is module-finite over R. A finitely generated algebra R over a field K is F-finite precisely when $K^{1/p}$ is a finite field extension of K.

We shall denote by R° the complement of the union of the minimal primes of R. An ideal $I = (x_1, \dots, x_n) \subseteq R$ is said to be a *parameter ideal* if the images of $x_1, \dots, x_n$ form part of a system of parameters in the local ring R_P for every prime ideal P containing I.

We shall denote by F the Frobenius endomorphism of R and by F^e, its eth iteration. For an ideal $I = (x_1, \dots, x_n) \subseteq R$, we use $I^{[q]}$ to denote $F^e\,(I)R = (x_1^q, \dots, x_n^q)$ where $q = p^e$. For an R-module M, the R-module structure of $F^e(M)$ is given by $r'(r \otimes m) = r'r \otimes m$, and $r' \otimes rm = r'r^q \otimes m$. For R-modules $N \subseteq M$, we use $N_M^{[q]}$ to denote $\mathrm{Im}(F^e(N) \to F^e(M))$. We say that $u \in N_M^*$, the *tight closure* of N in M, if there exists $c \in R^\circ$ such that $cu^q \in N_M^{[q]}$ for all $q = p^e \gg 0$. If $N_M^* = N$ we say that N is a *tightly closed* submodule of M.

It is worth recording the case when $M = R$ and $N = I$ is an ideal of R. For an element x of R, we say that $x \in I^*$ if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$.

For R-modules $N \subseteq M$ we define N_M^{*fg}, the *finitistic tight closure* of N in M, as the union of the modules $(M' \cap N)^*_{M'}$ where the union is taken over all the finitely generated submodules M' of M.

A ring R is *weakly F-regular* if every ideal of R is tightly closed, and is *F-regular* if every localization is weakly F-regular. An *F-rational* ring is one in which all parameter ideal are tightly closed.

The *test ideal* of the ring R is defined as

$$\tau_R = \bigcap_M \mathrm{Ann}_R 0_M^*,$$

where M runs through all finitely generated R-modules. If a local ring (R,m) is approximately Gorenstein, i.e., has a sequence of m-primary irreducible ideals $\{I_t\}$ cofinal with the powers of m, then

$$\tau_R = \bigcap_t (I_t : I_t^*).$$

It should be mentioned that an excellent reduced ring is approximately Gorenstein.

We say that an element $c \in R^{\circ}$ is a *test element* if for all ideals I of R, we have $cI^* \subseteq I$.

It is not known if the test ideal commutes with localization and completion in general, but several strong positive results are now available: in [10] Smith showed that if R is a complete local Gorenstein ring with test ideal τ, then the test ideal of a localization R_P is τR_P. In [2] Aberbach and MacCrimmon showed that the same result is true if R is a reduced $\mathbb{Q}$-Gorenstein local ring. More recently, Lyubeznik and Smith proved that if a ring R is Cohen-Macaulay with only isolated non-Gorenstein points, or if it has only isolated singularities, or if it is $\mathbb{N}$-graded over a field, then test elements of R continue to be test elements after localization and completion, see [8].

Our references for the theory of tight closure are [3–6]. We next record some well known facts about local cohomology and canonical modules.

Let (R,m,K) be a local ring of dimension with a system of parameters $x_1,\ldots,x_d$. The local cohomology module $H_m^d(R)$ of R may be identified with

$$\varinjlim \frac{R}{(x_1^t,\ldots,x_d^t)R},$$

where the maps in the direct are induced by multiplication by the element $x_1\cdots x_d$. Let $E_R = E_R(K)$ denote the injective hull of the residue field K as an R-module. If R is a complete local ring, then we may define the canonical module of R to be

$$\omega_R = \mathrm{Hom}_R(H_m^d(R), E_R)$$

and we then have

$$\mathrm{Hom}_R(\omega_R, E_R) \cong H_m^d(R).$$

If, in addition, R is normal then $H_m^d(\omega_R) \cong E_R$.

For a local inclusion $(R,m,K) \hookrightarrow (S,n,L)$ where S is module-finite over the subring R, the module $\mathrm{Hom}_R(S,\omega_R)$ is isomorphic to the canonical module of the ring S.

3. The module of relative test elements

Throughout this paper (R,m,K) and (S,n,L) will be complete local rings where S is module-finite over the subring R. We shall also assume that these rings are reduced and F-finite. In this setting we shall define the module of relative test elements as a submodule of $\mathrm{Hom}_R(S,R)$, the S-module consisting of R-linear homomorphism from S to the ring R.

Let E_R denote the injective hull of the residue field K as an R-module. Then $E_S = \mathrm{Hom}_R(S,E_R)$ is the injective hull of L as an S-module. The following isomorphisms are easily verified:

$$\begin{aligned}\mathrm{Hom}_S(E_R \otimes_R S, \mathrm{Hom}_R(S, E_R)) &\cong \mathrm{Hom}_R(E_R \otimes_R S \otimes_S S, E_R)\\ &\cong \mathrm{Hom}_R(E_R \otimes_R S, E_R)\\ &\cong \mathrm{Hom}_R(S, \mathrm{Hom}_R(E_R, E_R))\\ &\cong \mathrm{Hom}_R(S, R).\end{aligned}$$

Under these isomorphism, a map $\phi \in \mathrm{Hom}_R(S,R)$ corresponds to the map $\tilde{\phi} \in \mathrm{Hom}_S(E_R \otimes_R S, \mathrm{Hom}_R(S, E_R))$ where, for elements s and $s' \in S$ and $e \in E_R$, we have

$$(\tilde{\phi}(e \otimes s))(s') = \phi(ss')e \in E_R.$$

Consider the inclusion

$$0^{*fg}_{E_R \otimes_R S} \subseteq E_R \otimes_R S,$$

where $0^{*fg}_{E_R \otimes_R S}$ denotes the finitistic tight closure of the zero submodule of the S-module, $E_R \otimes_R S$. Applying the functor $^\vee = \mathrm{Hom}_S(\ , E_S)$, we get a surjection

$$\mathrm{Hom}_R(S,R) \twoheadrightarrow (0^{*fg}_{E_R \otimes_R S})^\vee.$$

and the *module of relative test elements* $T(S,R)$ is defined as the kernel of this surjection:

$$T(S,R) = \{\phi \in \mathrm{Hom}_R(S,R)\colon \tilde{\phi}(0^{*fg}_{E_R \otimes_R S}) = 0\}.$$

Proposition 1. *For complete local reduced rings R and S where S is module-finite over the subring R, the following are equivalent characterizations of the module of relative test elements*:
(1) $T(S,R) = \{\phi \in \mathrm{Hom}_R(S,R)\colon \tilde{\phi}(0^{*fg}_{E_R \otimes_R S}) = 0\}$,
(2) $T(S,R) = \{\phi \in \mathrm{Hom}_R(S,R)\colon \phi((IS)^*) \subseteq I$ for all ideals $I \subseteq R\}$, *and*
(3) $T(S,R) = \{\phi \in \mathrm{Hom}_R(S,R)\colon \phi((I_tS)^*) \subseteq I_t$ *where* $\{I_t\}$ *is a sequence of m-primary irreducible ideals of R cofinal with the powers of m*.

Proof. Since R is an excellent reduced ring, it is approximately Gorenstein, and there exists a sequence $\{I_t\}$ of m-primary irreducible ideals cofinal with the powers of the maximal ideal. Since $R/I_t \cong \mathrm{Ann}_{E_R}(I_t)$, we may write $E_R = \varinjlim R/I_t$. Note that

$$(\tilde{\phi}(s \bmod I_t))(s') = \phi(ss') \bmod I_t.$$

The condition $\tilde{\phi}(0^{*fg}_{E_R \otimes_R S}) = 0$ is therefore equivalent to the condition that $\phi((I_tS)^*) \subseteq I_t$ for all $t \geq 1$. It remains to show that this implies $\phi((IS)^*) \subseteq I$ for all ideals $I \subseteq R$.

Let $\phi \in \mathrm{Hom}_R(S,R)$ be a homomorphism which satisfies $\phi((I_tS)^*) \subseteq I_t$ for all $t \geq 1$. If there exists an element $x \in R$ and an ideal I such that $x \in (IS)^*$ but $\phi(x) \notin I$ then we may choose I to be maximal with respect to this property, i.e., $\phi(x) \in J$ for every ideal J strictly bigger than I. Note that this implies $\phi(x)m \subseteq I$ and consequently that $R\phi(x) \cong R/m$. In other words, the R/I-linear inclusion

$$R/m \to R/I \quad \text{where} \quad 1 \mapsto \overline{\phi(x)}$$

is an essential extension. It follows that I must be m-primary and so we may pick t such that $I_t \subseteq I$. Let $r \in R$ be the preimage of a socle generator in R/I_t. There is a homomorphism $R/m \to R/I_t$ under which $1 \mapsto \bar{r}$. Since R/I_t is injective when regarded as a module over itself, we get a homomorphism $\alpha : R/I \to R/I_t$. Let $\alpha(1) = \bar{\lambda} \in R/I_t$ where $\lambda \in R$.

$$\begin{tikzcd} 0 \arrow[r] & R/m \arrow[r, "1\mapsto\overline{\phi(x)}"] \arrow[d, "1\mapsto\bar{r}"'] & R/I \\ & R/I_t & \end{tikzcd} .$$

Since $\lambda I \subseteq I_t$ and $x \in (IS)^*$, we get $\lambda x \in (I_t S)^*$. By our hypothesis on ϕ, $\phi(\lambda x) = \lambda\phi(x) \in I_t$. However, the commutativity of the above diagram tells us that $\overline{\lambda\phi(x)} = \bar{r} = 0$ in R/I_t, a contradiction since $\bar{r}$ is the socle generator in R/I_t. $\square$

It follows immediately from the above proposition that for a complete local reduced ring S, after identifying $\mathrm{Hom}_S(S,S)$ with the ring S, the module $T(S,S)$ is precisely the test ideal of S. We next record some observations about the module of relative test elements:

Proposition 2. *Let R and S be complete local reduced rings where the ring S is module-finite over its subring R.*

(1) *If τ_S denotes the test ideal of S, then $\tau_S \mathrm{Hom}_R(S,R) \subseteq T(S,R)$.*
(2) *If $\phi \in T(S,R)$ then $\phi(1)$ is an element of τ_R, the test ideal of R.*
(3) *Assume that R is a direct summand of S as an R-module. Then $c \in \tau_R$ if and only if there exists $\phi \in T(S,R)$ with $\phi(1) = c$.*
(4) *The ring R is weakly F-regular if and only if there exists an element $\phi \in T(S,R)$ with $\phi(1) = 1$.*

Proof. (1) Let $c \in \tau_S$ and $\phi \in \mathrm{Hom}_R(S,R)$. If $x \in (IS)^*$ for an ideal I of R, then $cx \in IS$ and consequently

$$(c\phi)(x) = \phi(cx) \in \phi(IS) \subseteq I.$$

Hence $c\phi \in T(S,R)$.

(2) If $x \in I^*$ then $x \in (IS)^*$ and so for $\phi \in T(S,R)$ we have $\phi(1)x = \phi(x) \in I$. Consequently $\phi(x) \in \tau_R$.

(3) If R is a direct summand of S, we have an inclusion $i : E_R \hookrightarrow E_R \otimes_R S$ where $i : e \mapsto e \otimes 1$. Applying the functor ${}^\vee = \mathrm{Hom}_R(\ , E_R)$, we get a map $\mathrm{Hom}_R(S,R) \to R$ and it is easily verified that this map takes a homomorphism $\phi \in \mathrm{Hom}_R(S,R)$ to the element $\phi(1) \in R$. We first show that

$$0^{*fg}_{E_R \otimes_R S} \cap i(E_R) \subseteq i(0^{*fg}_{E_R}).$$

To see this, choose $d \in R$ which is a test element for the ring R as well as for the ring S. Let $u \otimes 1 \in 0^*_{N \otimes_R S} \cap i(N)$ where N is a finitely generated submodule of E_R which contains the element u. Then

$$d(u)^q = 0 \quad \text{in} \quad F^e_S(N \otimes_R S)$$

and so $u \otimes 1 \in i(0^*_N)$.

If an element $c \in R$ is in τ_R, then $c \in \mathrm{Ann}_R(0^{*fg}_{E_R})$, and so

$$c \in \mathrm{Ann}_R(0^{*fg}_{E_R \otimes_R S} \cap i(E_R)).$$

Consequently, c kills the kernel of the homomorphism

$$0^{*fg}_{E_R \otimes_R S} \oplus E_R \xrightarrow{(1_{0^*}, -i)} E_R \otimes_R S.$$

Applying the functor ${}^\vee = \mathrm{Hom}_S(\ , E_S)$ we see that c kills the cokernel of the homomorphism

$$\mathrm{Hom}_R(S, R) \xrightarrow{(1_{0^*}, -i)^\vee} (0^{*fg}_{E_R \otimes_R S})^\vee \times R.$$

Hence, there exists a homomorphism $\phi \in \mathrm{Hom}_R(S, R)$ with

$$(\phi \bmod T(S, R), \phi(1)) = c(0, 1),$$

i.e., $\phi \in T(S, R)$ and $\phi(1) = c$.

(4) If R is weakly F-regular, then it is a direct summand of the module-finite extension ring S by Theorem 5.25 of Hochster and Huneke [5]. The result then follows from (2) and (3) above. □

4. The case of a Gorenstein subring

We examine the case when the subring R is assumed to be Gorenstein. This hypothesis gives us an isomorphism $\mathrm{Hom}_R(S, R) \cong \omega_S$, where ω_S is the canonical module of S. Note that

$$E_R \otimes_R S \cong H^d_m(R) \otimes_R S \cong H^d_n(S),$$

where d is the dimension of the ring S. Let $x_1, \ldots, x_d$ be a system of parameters for the ring S. If the ring S is normal then

$$H^d_n(\omega_S) = \varinjlim \frac{\omega_S}{(x_1^t, \ldots, x_d^t)\omega_S}$$

is isomorphic to E_S, the injective hull of the residue field of S. There is a natural action of ω_S on $H^d_n(S)$ given by the pairing $\omega_S \times H^d_n(S) \to E_S$. Taking ω_S to be $\mathrm{Hom}_R(S, R)$, we have

$$T(S, R) = \mathrm{Ker}(\omega_S \twoheadrightarrow (0^*_{H^d_n(S)})^\vee) = \mathrm{Ann}_{\omega_S} 0^*_{H^d_n(S)}.$$

Note that as a submodule of a fixed canonical module ω_S, the module $T(S, R)$ does not depend on the choice of the Gorenstein subring R and in fact $T(S, R)$ is precisely the *parameter test module* defined in [11]. We shall say that for a complete local normal domain S, the parameter test module is

$$T_G(S) = \mathrm{Ann}_{\omega_S} 0^*_{H^d_n(S)}.$$

In the discussion above, we have established:

Proposition 3. *Let $(R,m,K) \subseteq (S,n,L)$ be complete local rings of dimension d where the ring S is Cohen–Macaulay and normal and is module-finite over the Gorenstein subring R. Then, the module $T(S,R)$ of relative test elements, as a submodule of the canonical module of S, is $T_G(S) = \mathrm{Ann}_{\omega_S} 0^*_{H^d_n(S)}$.*

Since $0^*_{H^d_n(S)}$ is a submodule of $H^d_n(S)$ stable under the action of the Frobenius, in the terminology of Smith [11] its annihilator $T_G(S)$ in ω_S is an *F-submodule* of ω_S. Using the results of Smith [11] we see that $T_G(S)$ has suitable localization properties.

Proposition 4. *Let S be a complete local domain of dimension d which is Cohen–Macaulay and normal. If P is a prime ideal of the ring S, then*

$$T_G(S) \otimes_S S_P = T_G(S_P).$$

Proof. If $c \in T_G(S)$ we first show that $c/1 \in T_G(S_P) = \mathrm{Ann}_{\omega_{S_P}} 0^*_{H^i(S_P)}$. Let the prime ideal P have height i. If

$$\left[\frac{z}{1} + \left(\frac{x_1^t}{1}, \ldots, \frac{x_i^t}{1}\right)\right] \in 0^*_{H^i(S_P)}, \quad \text{we then have} \quad \frac{z}{1} \in \left(\left(\frac{x_1^t}{1}, \ldots, \frac{x_i^t}{1}\right) S_P\right)^*.$$

By Aberbach et al. [1, Theorem 6.9], tight closure commutes with localization for ideals generated by a regular sequence, and so there exists an element $u \in S - P$ such that $uz \in (x_1^t, \ldots, x_i^t)S^*$. Hence, for all $H \geq 1$, we have $uz \in (x_1^t, \ldots, x_i^t, x_{i+1}^H, \ldots, x_d^H)S^*$. Since $c \in T_G(S)$, this gives us

$$cuz \in (x_1^t, \ldots, x_i^t, x_{i+1}^H, \ldots, x_d^H)\omega_S \quad \text{for all } H \geq 1,$$

and consequently that $cuz \in (x_1^t, \ldots, x_i^t)\omega_S$. Hence,

$$\frac{c}{1} \times \left[\frac{z}{1} + \left(\frac{x_1^t}{1}, \ldots, \frac{x_i^t}{1}\right) S_P\right] = 0 \quad \text{under the pairing} \quad \omega_{S_P} \times H^i(S_P) \to E_{S_P}.$$

To show the inclusion $T_G(S_P) \subseteq T_G(S) \otimes_S S_P$, we first note that by Smith [11, Lemma 2.1] we have

$$T_G(S) \otimes_S S_P = \mathrm{Ann}_{\omega_{S_P}}(\mathrm{Hom}_{S_P}(\mathrm{Hom}_S(0^*_{H^d_n(S)}, E_S), E_S(S/P))).$$

Hence, it suffices to establish the inclusion

$$\mathrm{Hom}_{S_P}(\mathrm{Hom}_S(0^*_{H^d_n(S)}, E_S), E_S(S/P)) \subseteq 0^*_{H^i(S_P)}.$$

This is proved in [11, Equation 5.2.2]. □

We next obtain a corollary analogous to Smith [11, Proposition 6.1].

Corollary 5. *Let S be a complete local domain which is Cohen–Macaulay and normal, and let $x_1,\ldots,x_d$ be a system of parameters for S. If we consider the module of relative test elements $T_G(S)$ as a submodule of the canonical module ω, then $T_G(S)$ is not contained in $(x_1,\ldots,x_d)\omega$. In particular if S is Gorenstein, the test ideal is not contained in any parameter ideal.*

Proof. The ring S is module finite over the regular ring $K[[x_1,\ldots,x_d]]$, and so by Proposition 2, there exists $\phi \in T_G(S)$ with $\phi(1) = 1$. But this means that $\phi \notin (x_1,\ldots,x_d)\omega$. If S is Gorenstein, recall that we may identify $T_G(S) \subseteq \omega$ with $\tau_S \subseteq S$, and the result follows. □

Proposition 2(1) shows that $\tau_S\omega_S \subseteq T_G(S)$. If S is an F-rational ring which is not F-regular, then $T_G(S) = \omega_S$ and so $\tau_S\omega_S \subsetneq T_G(S)$. If the ring S is Gorenstein then we may take ω_S and we have $T_G(S) = \tau_S$, as is recorded in Proposition 7. We next present an example below where although the ring S is not Gorenstein, it turns out that $\tau_S\omega_S = T_G(S)$.

Example 6. Let $A = K[[X,Y,Z]]/(X^4 + Y^4 + Z^4)$ where the characteristic of the field K is a prime integer $p \geq 3$. Using lower-case letters to denote corresponding images, let S be the subring of T generated by the elements $x^2, xy, xz, y^2, yz, z^2$. The ring S is not F-rational and $0^*_{H^2(S)}$ is spanned by the following three elements of $H^2_n(S)$,

$$\eta_1 = [z^2xy + (x^2, y^2)S],\ \eta_2 = [z^3x + (x^2, y^2)S] \text{ and } \eta_3 = [z^3y + (x^2, y^2)S].$$

It is not hard to see that the ideal $\omega_S = (x^2, xy, xz)S$ is a canonical module for S – the corresponding computations in the graded case can be performed using the fact that the canonical module ω_S of the Veronese subring $S = T^{(2)}$ is the Veronese submodule $\omega_T^{(2)}$, where ω_T is the graded canonical module of T. We shall compute $T_G(S)$ as an ideal contained in $(x^2, xy, xz)S$.

The test ideal of A is $m_A^2 = (x^2, xy, xz, y^2, yz, z^2)A$ and since the ring S is a direct summand of A, it is easy to verify that its test ideal is $\tau_S = (x^2, xy, xz, y^2, yz, z^2)S$.

A routine computation shows that

$$\mathrm{Ann}_{\omega_S}\eta_1 = (x^2, xy, xz^3), \quad \mathrm{Ann}_{\omega_S}\eta_2 = (x^2, xz, xy^3)$$

and $\mathrm{Ann}_{\omega_S}\eta_3 = (xy, xz, x^4)$. Consequently, we have

$$T_G(S) = (x^4, x^3y, x^3z, x^2y^2, x^2yz, x^2z^2, xy^3, xy^2z, xyz^2, xz^3) = \tau_S\omega_S.$$

Proposition 7. *Let (S, n, L) be complete local normal Gorenstein ring. Then $T_G(S)$ is isomorphic to the test ideal τ_S.*

Proof. Since S is Gorenstein, we have an isomorphism $\omega_S \cong S$ and consequently

$$T_G(S) \cong \mathrm{Ann}_S 0^{*fg}_{H^d_n(S)} = \mathrm{Ann}_S 0^{*fg}_{E_S} = \tau_S. \qquad \square$$

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