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Anurag K. Singh

Separable integral extensions and plus closure

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Abstract. We show that an excellent local domain of characteristic p has a separable big Cohen–Macaulay algebra. In the course of our work we prove that an element which is in the Frobenius closure of an ideal can be forced into the expansion of the ideal to a module-finite separable extension ring.

1. Introduction

Let R be an excellent domain of characteristic p, and let R^+ denote the integral closure of R in an algebraic closure of its fraction field. A celebrated theorem of M. Hochster and C. Huneke states that R^+ is a big Cohen–Macaulay algebra for R, see [HH2]. The *plus closure* I^+ of an ideal I of R is defined as $I^+ = IR^+ \cap R$, and has close connections with tight closure theory: the containment $I^+ \subseteq I^*$ is easily verified, and in [Sm] K. E. Smith showed that $I^+ = I^*$ if the ideal I is generated by part of a system of parameters for R. In this paper we establish the somewhat surprising result that for any element $z \in I^+$ there exists an integral domain S, which is a *separable* module–finite extension of R, such that $z \in IS$. We use this idea to obtain a separable big Cohen–Macaulay algebra $R^{+\text{sep}}$ for any excellent local domain (R, m) of characteristic p.

For related work on R^+ and plus closure see [Ab2] and [AH]. Our references for the theory of tight closure are [HH1], [HH3] and [HH4].

In Section 4 we present an explicit computation of plus closure, specifically we show that $xyz \in (x^2, y^2, z^2)^+$ in the cubic hypersurface $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$. This strengthens a result obtained in [Si2] where it was established that $xyz \in (x^2, y^2, z^2)^*$.

A. K. Singh: Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, USA.
e-mail: singh6@math.uiuc.edu

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2. Preliminaries

Let R be a Noetherian ring of characteristic $p > 0$. The letter e shall denote a variable nonnegative integer, and q shall denote the corresponding power of the characteristic, i.e., $q = p^e$. For an ideal $I = (x_1, \ldots, x_n) \subseteq R$, let $I^{[q]} = (x_1^q, \ldots, x_n^q)$. For a reduced ring R of characteristic $p > 0$, $R^{1/q}$ shall denote the ring obtained by adjoining all q th roots of elements of R.

For an element x of R we say that $x \in I^F$, the *Frobenius closure* of I, if there exists an integer $q = p^e$ such that $x^q \in I^{[q]}$. The ring R is said to be *F-pure* if the Frobenius homomorphism is pure, i.e., if $F : M \to F(M)$ is injective for all R-modules M.

In specific examples which are homomorphic images of polynomial rings, we shall use lower case letters to denote the images of the corresponding variables.

3. Separable integral extensions

Our main result regarding Frobenius closure and separable extensions is the following theorem:

Theorem 3.1. *Let R be a excellent domain of characteristic $p > 0$, $I \subseteq R$ an ideal, and $z \in R$ an element such that $z \in I^F$. Then there exists an integral domain S, which is a module-finite separable extension of R, such that $z \in IS$.*

Proof. Since $z \in I^F$, there exists a positive integer $q = p^e$, nonzero elements $x_0, \ldots, x_n \in I$, and $a_0, \ldots, a_n \in R$ such that

$$z^q = \sum_{i=0}^{n} a_i x_i^q.$$

For $1 \leq i \leq n$, consider the equations

$$U_i^q + U_i x_0^q - a_i = 0.$$

These are monic separable equations in the variables U_i, and therefore have solutions u_i in a separable field extension of the fraction field of R. Let S be the integral closure of the ring $R[u_1, \ldots, u_n]$ in its field of fractions. We shall show that $z \in IS$. Let

$$u_0 = (z - \sum_{i=1}^{n} x_i u_i)/x_0,$$

which is an element of the fraction field of S. Taking the q th power of this element, we have

$$\begin{aligned} u_0^q &= (z^q - \sum_{i=1}^{n} x_i^q u_i^q)/x_0^q = (\sum_{i=0}^{n} a_i x_i^q - \sum_{i=1}^{n} x_i^q u_i^q)/x_0^q \\ &= a_0 + \sum_{i=1}^{n} (a_i - u_i^q) x_i^q / x_0^q = a_0 + \sum_{i=1}^{n} u_i x_i^q. \end{aligned}$$

Consequently u_0 is integral over S, but since S is a normal domain, we then have $u_0 \in S$. This implies that $z = \sum_{i=0}^{n} x_i u_i \in IS$. □

Remark 3.2. Let $GL_n(\mathbb{F}_q)$ be the general linear group over the finite field $\mathbb{F}_q$. In [Si1] the author constructed examples to show that the ring of invariants for the natural action of a subgroup G of $GL_n(\mathbb{F}_q)$ on the polynomial ring $\mathbb{F}_q[X_1, \ldots, X_n]$ need not be F–pure. Specifically, consider the natural action of the symplectic group $Sp_4(\mathbb{F}_q)$ on the polynomial ring $\mathbb{F}_q[X_1, X_2, X_3, X_4]$. Then the ring of invariants is isomorphic to the hypersurface

$$R = \mathbb{F}_q[X, Y, Z, A, B]/(Z^q - AX^q - BY^q)$$

which is not F-pure since $z \in I = (x, y)^F$, more precisely the element z is forced into the expanded ideal $IR^{1/q}$ in the purely inseparable extension $R^{1/q}$. However z is also forced into the expansion of I to the linearly disjoint separable extension $\mathbb{F}_q[X_1, X_2, X_3, X_4]$. These examples provided the basic case of the equational construction used in the proof of Theorem 3.1 above.

Definition 3.3. *Let R be an integral domain with fraction field $\mathcal{K}$. Let $\overline{\mathcal{K}}$ be an algebraic closure of the field $\mathcal{K}$, and $\mathcal{L}$ be the maximal separable field extension of $\mathcal{K}$ in $\overline{\mathcal{K}}$. Then $R^{+\text{sep}}$ shall denote the integral closure of R in $\mathcal{L}$.*

With this notation, Theorem 3.1 gives us the following corollary:

Corollary 3.4. *Let R be a excellent integral domain of characteristic p. If I is an ideal of R and z an element of R such that $z \in I^+$, then there exists a module-finite separable extension domain S such that $z \in IS$. Consequently for all ideals I of R we have*

$$IR^+ \cap R = IR^{+\text{sep}} \cap R.$$

Proof. Since $z \in I^+$ there exists a integral domain R_1, module-finite over R, such that $z \in IR_1$. If R_2 denotes the largest separable extension of R contained in R_1, we have $z \in (IR_2)^F$. By Theorem 3.1 there exists a separable module-finite extension S of R_2 such that $z \in IS$. □

Let $R = \oplus_{i\in\mathbb{N}} R_i$ be an $\mathbb{N}$-graded integral domain over a field $K = R_0$ of characteristic p. An element $z \in R^+$ is said to be *homogeneous* if it satisfies an equation of integral dependence over R of the form

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

where $a_i \in R_{id}$ for some $d \in \mathbb{Q}$, for all $1 \leq i \leq n$. (The element z can then be assigned weight d, and satisfies a homogeneous equation of integral dependence over R.) Let $R^{+\mathrm{GR}}$ denote the $\mathbb{Q}$-graded subalgebra of R^+ which is generated by homogeneous elements $z \in R^+$. In this setting, Hochster and Huneke show that $R^{+\mathrm{GR}}$ is a graded big Cohen–Macaulay algebra for R, i.e., that every homogeneous system of parameters for R is a regular sequence on $R^{+\mathrm{GR}}$, see [HH2].

The following example shows that for R as above, a homogeneous ideal I of R, and a homogeneous element $\xi \in I^+$, there need not exist a graded separable module-finite extension S with $\xi \in IS$.

Example 3.5. Let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$ where K is an algebraically closed field of characteristic 2. Note that $z^2 \in (x, y)^F$ since $z^4 = zx^3 + zy^3$. We claim R has no module-finite graded separable extension S with $z^2 \in IS$. If S were such an extension, there exist homogeneous elements $u, v \in S$ (of weight 1) with $z^2 = ux + vy$. In $R^{+\mathrm{GR}}$ we then have

$$z^2 = ux + vy = x\sqrt{xz} + y\sqrt{yz},$$

and since $R^{+\mathrm{GR}}$ is a graded big Cohen–Macaulay algebra for R, the relation $x(u + \sqrt{xz}) = y(v + \sqrt{yz})$ must be trivial in $R^{+\mathrm{GR}}$. Hence there exists $c \in R^{+\mathrm{GR}}$ such that $u + \sqrt{xz} = cy$ and $v + \sqrt{yz} = cx$, but then c must have weight 0, and so satisfies an equation of integral dependence over the algebraically closed field K. Consequently $c \in K$, and therefore $\sqrt{xz}$ and $\sqrt{yz}$ are elements of a separable extension of R, a contradiction.

Remark 3.6. Let R be a ring of characteristic p which is not F-pure, i.e., such that there exists $z \in R$ with $z \in I^F - I$. By Theorem 3.1 there exists a module-finite separable extension S such that $z \in IS$. In general we cannot expect S to be F-regular or even F-rational: if R is the coordinate ring of an elliptic curve, a normal module-finite extension domain S will continue to have elements of infinite order in the divisor class group $\mathrm{Cl}(S)$, and consequently the two dimensional ring S cannot be F-rational by a result of J. Lipman, [Li]. However there do exist examples where the separable extension is F-regular, moreover is a polynomial ring.

Example 3.7. Let $R = K[X, Y, Z, A, B]/(Z^q - AX^q - BY^q)$ where K is a field of characteristic p and $q = p^e$. Then $z \in (x, y)^F$, and we construct

a separable module-finite extension S such that $z \in (x, y)S$. Let u be a root of the separable equation

$$U^q - a + Uy^q = 0$$

and let S be the normalization of $R[u]$. Then $v = (z - ux)/y$ is easily seen to be an element of S and we have $S = K[X, Y, u, v]$, which is a polynomial ring.

We next recall a definition from [Ma]:

Definition 3.8. *A Noetherian integral domain R is said to be a splinter if it is a direct summand, as an R-module, of every module-finite extension ring of R.*

In the case that R contains the field of rational numbers, it is easily seen that R is a splinter if and only if it is a normal ring, but the notion is more subtle for rings of characteristic p. F-regular rings are always splinter and the converse is known to hold for $\mathbb{Q}$-Gorenstein rings, see [Si3]. Corollary 3.4 above gives a new characterization of splinter rings of characteristic p:

Corollary 3.9. *Let R be an excellent integral domain on characteristic p. Then R is a splinter if and only if it is a direct summand of every module-finite separable extension domain.*

Proof. Excellent integral domains are approximately Gorenstein, and so an inclusion $R \to S$ splits if and only if $IS \cap R = I$ for all ideals I of R, see [Ho]. The result now follows from Corollary 3.4. □

We also obtain the following theorem which is separable analogue of the main result of [HH2].

Theorem 3.10. *Let (R, m) be an excellent local domain of characteristic p. Then every sequence of elements which is part of a system of parameters for R is a regular sequence on $R^{+\text{sep}}$. Consequently $R^{+\text{sep}}$ is a balanced big Cohen–Macaulay module for R.*

Proof. Let $x_1, \ldots, x_k$ be part of a system of parameters for R. Given a relation $\sum_{i=1}^{k} r_i x_i = 0$ with $r_i \in R^{+\text{sep}}$ we may replace R by a module-finite separable extension and (after a change of notation) assume that $r_i \in R$. Since R^+ is a big Cohen–Macaulay algebra for R, [HH2, Theorem 1.1], we have $r_k \in (x_1, \ldots, x_{k-1})R^+ \cap R$, but then $r_k \in (x_1, \ldots, x_{k-1})R^{+\text{sep}}$ by Corollary 3.4. □

4. A computation of plus closure

In [Si2] we showed that $xyz \in (x^2, y^2, z^2)^*$ in the cubic hypersurface $R = K[X, Y, Z]/(X^3+Y^3+Z^3)$ where K is a field of prime characteristic $p \neq 3$. This question arose in M. McDermott's study of the tight closure and plus of various irreducible ideals in R, see [Mc]. We furthermore showed that $xyz \in (x^2, y^2, z^2)^F$ whenever R is not F-pure, i.e., when $p \equiv 2 \mod 3$. In this section we settle the one unresolved issue by establishing that $xyz \in (x^2, y^2, z^2)^+$ when the characteristic of the field is $p \equiv 1 \mod 3$.

The following lemma is a basic version of a more powerful equational criterion, but will suffice for our needs.

Lemma 4.1. *Let R be a integral domain of characteristic p and consider nonzero elements $z, x_1, \ldots, x_k \in R$ which satisfy*

$$z^p \in (x_1^p, \ldots, x_k^p) + z((x_1^p, \ldots, x_k^p) : (x_1, \ldots, x_k)).$$

Then there exists an integral domain S which is a module-finite extension of R such that $z \in (x_1, \ldots, x_k)S$.

Proof. See [Ab1]. □

We record a determinant identity that we shall find useful. For integers n and m where $m \geq 1$, we shall use the notation:

$$\binom{n}{m} = \frac{(n)(n-1)\cdots(n-m+1)}{(m)(m-1)\cdots(1)}.$$

Lemma 4.2.

$$\det \begin{pmatrix} \binom{n}{a+k} & \binom{n}{a+k+1} & \cdots & \binom{n}{a+2k} \\ \binom{n}{a+k-1} & \binom{n}{a+k} & \cdots & \binom{n}{a+2k-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \binom{n}{a} & \binom{n}{a+1} & \cdots & \binom{n}{a+k} \end{pmatrix} = \frac{\binom{n}{a+k}\binom{n+1}{a+k}\cdots\binom{n+k}{a+k}}{\binom{a+k}{a+k}\binom{a+k+1}{a+k}\cdots\binom{a+2k}{a+k}}.$$

Proof. This is evaluated in [Mu, page 682] as well as [Ro]. □

As a first application of this, we prove the following lemma:

Lemma 4.3. *Let $K[A, B]$ be a polynomial ring over a field K of characteristic $p = 3k + 1$, where k is a positive integer. Then*

$$(A, B)^{3k} \subseteq I = (A^{2k+1},\ B^{2k+1},\ (A + B)^{2k}).$$

Proof. Note that I contains the following elements:

$$(A+B)^{2k}A^k,\ (A+B)^{2k}A^{k-1}B,\ldots,\ (A+B)^{2k}B^k.$$

We consider the binomial expansions of these elements modulo the ideal (A^{2k+1}, B^{2k+1}). Consequently the following elements are in I :

$$\begin{aligned}
&\tbinom{2k}{k}A^{2k}B^k + \tbinom{2k}{k+1}A^{2k-1}B^{k+1} + \cdots + \tbinom{2k}{2k}A^kB^{2k},\\
&\tbinom{2k}{k-1}A^{2k}B^k + \tbinom{2k}{k}A^{2k-1}B^{k+1} + \cdots + \tbinom{2k}{2k-1}A^kB^{2k},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\tbinom{2k}{0}A^{2k}B^k + \tbinom{2k}{1}A^{2k-1}B^{k+1} + \cdots + \tbinom{2k}{k}A^kB^{2k}.
\end{aligned}$$

The coefficients of $A^{2k}B^k,\ A^{2k-1}B^{k+1},\ldots,\ A^kB^{2k}$ form the matrix:

$$\begin{pmatrix}
\binom{2k}{k} & \binom{2k}{k+1} & \cdots & \binom{2k}{2k}\\
\binom{2k}{k-1} & \binom{2k}{k} & \cdots & \binom{2k}{2k-1}\\
\cdots\cdots\cdots\cdots & & & \\
\cdots\cdots\cdots\cdots & & & \\
\binom{2k}{0} & \binom{2k}{1} & \cdots & \binom{2k}{k}
\end{pmatrix}.$$

To show that all monomials of degree $3k$ in A and B are in I, it suffices to show that this matrix is invertible. By Lemma 4.2, the determinant of this matrix is

$$\frac{\binom{2k}{k}\binom{2k+1}{k}\cdots\binom{3k}{k}}{\binom{k}{k}\binom{k+1}{k}\cdots\binom{2k}{k}}$$

which is immediately seen to be invertible since the characteristic of the field is $p = 3k+1$. □

Lemma 4.4. *Let $R = K[X,Y,Z]/(X^3+Y^3+Z^3)$ where K is a field of prime characteristic $p = 3k+1$. Then*

$$x^{p-1}y^{2p-2} \in (x^{2p}, y^{2p}, z^{2p}) : (x^2, y^2, z^2).$$

Proof. We first show that $x^{p+1}y^{2p-2} \in (x^{2p}, y^{2p}, z^{2p})$, i.e., that

$$x^{3k+2}y^{6k} \in (x^{6k+2},\ y^{6k+2},\ z^{6k+2}).$$

This would follow if we establish that

$$x^{3k}y^{6k} \in (x^{6k},\ y^{6k+3},\ z^{6k+3}).$$

Setting $A = y^3$, $B = z^3$ and $A + B = -x^3$, we need to show

$$(A + B)^k A^{2k} \in I = ((A + B)^{2k},\ A^{2k+1},\ B^{2k+1}).$$

This follows immediately from Lemma 4.3.

It remains to check that $x^{p-1}y^{2p-2}z^2 \in (x^{2p}, y^{2p}, z^{2p})$. This would follow if we show

$$x^{3k}y^{6k} \in (x^{6k+3},\ y^{6k+3},\ z^{6k}).$$

Setting $A = x^3$, $B = y^3$ and $A + B = -z^3$, we now need to show

$$A^k B^{2k} \in I = (A^{2k+1},\ B^{2k+1},\ (A + B)^{2k}),$$

which again follows from Lemma 4.3. □

We are now ready to prove our main result of this section.

Theorem 4.5. *Let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$ where K is a field of prime characteristic $p \equiv 1 \mod 3$. Then $xyz \in (x^2, y^2, z^2)^+$.*

Proof. By Lemma 4.1, it suffices to show

$$(xyz)^p \in (x^{2p}, y^{2p}, z^{2p}) + xyz((x^{2p}, y^{2p}, z^{2p}) : (x^2, y^2, z^2)).$$

By Lemma 4.4 we have

$$(x^{p-1}y^{2p-2},\ x^{2p-2}y^{p-1}) \subseteq (x^{2p}, y^{2p}, z^{2p}) : (x^2, y^2, z^2)),$$

and hence it is enough to show

$$(xyz)^p \in (x^{2p},\ y^{2p},\ z^{2p},\ x^p y^{2p-1}z,\ x^{2p-1}y^p z),$$

i.e., that

$$(xyz)^{3k+1} \in (x^{6k+2},\ y^{6k+2},\ z^{6k+2},\ x^{3k+1}y^{6k+1}z,\ x^{6k+1}y^{3k+1}z).$$

This follows if we show

$$(xyz)^{3k} \in (z^{6k+3},\ x^{3k}y^{6k},\ x^{6k}y^{3k}).$$

Setting $A = x^3$, $B = y^3$ and $A + B = -z^3$, we need to show

$$(AB(A + B))^k \in I = ((A + B)^{2k+1},\ A^k B^{2k},\ A^{2k}B^k).$$

Note that I contains the following elements:

$$(A + B)^{2k+1}A^{k-2}B,\ (A + B)^{2k+1}A^{k-3}B^2, \ldots,\ (A + B)^{2k+1}B^{k-1}.$$

We consider the binomial expansions of these elements modulo the ideal $(A^kB^{2k}, A^{2k}B^k)$. This shows that the following are elements of I :

$$\begin{aligned}
&\tbinom{2k+1}{k}A^{2k-1}B^{k+1} + \tbinom{2k+1}{k+1}A^{2k-2}B^{k+2} + \cdots + \tbinom{2k+1}{2k-2}A^{k+1}B^{2k-1},\\
&\tbinom{2k+1}{k-1}A^{2k-1}B^{k+1} + \tbinom{2k+1}{k}A^{2k-2}B^{k+2} + \cdots + \tbinom{2k+1}{2k-3}A^{k+1}B^{2k-1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\tbinom{2k+1}{2}A^{2k-1}B^{k+1} + \tbinom{2k+1}{3}A^{2k-2}B^{k+2} + \cdots + \tbinom{2k+1}{k}A^{k+1}B^{2k-1}.
\end{aligned}$$

The coefficients of $A^{2k-1}B^{k+1},\ A^{2k-2}B^{k+2}, \ldots,\ A^{k+1}B^{2k-1}$ form the matrix:

$$\begin{pmatrix}
\binom{2k+1}{k} & \binom{2k+1}{k+1} & \cdots & \binom{2k+1}{2k-2}\\
\binom{2k+1}{k-1} & \binom{2k+1}{k} & \cdots & \binom{2k+1}{2k-3}\\
\cdots\cdots\cdots\cdots\cdots & & & \\
\cdots\cdots\cdots\cdots\cdots & & & \\
\binom{2k+1}{2} & \binom{2k+1}{3} & \cdots & \binom{2k+1}{k}
\end{pmatrix}.$$

By Lemma 4.2, the determinant of this matrix is

$$\frac{\binom{2k+1}{k}\binom{2k+2}{k}\cdots\binom{3k-1}{k}}{\binom{k}{k}\binom{k+1}{k}\cdots\binom{2k-2}{k}}$$

which is invertible since the characteristic of the field is $p = 3k+1$. Hence all monomials of degree $3k$ in A and B are elements of the ideal I. □

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References

[Ab1] Aberbach, I.: *Finite phantom projective dimension and a phantom analogue of the Auslander-Buchsbaum theorem*. Thesis, University of Michigan, 1990

[Ab2] Aberbach, I.: Tight closure in F-rational rings. Nagoya Math. J. **135**, 43–54 (1994)

[AH] Aberbach, I. and Hochster, M.: Finite Tor dimension and failure of coherence in absolute integral closures. J. Pure Appl. Algebra **122**, 171–184 (1997)

[Ho] Hochster, M.: Cyclic purity versus purity in excellent Noetherian rings. Trans. Am. Math. Soc. **231**, 463–488 (1977)

[HH1] Hochster, M. and Huneke, C.: Tight closure, invariant theory, and the Briançon-Skoda theorem. J. Am. Math. Soc. **3**, 31–116 (1990)

[HH2] Hochster, M. and Huneke, C.: Infinite integral extensions and big Cohen-Macaulay algebras. Ann. Math., II. Ser. **135**, 53–89 (1992)

[HH3] Hochster, M. and Huneke, C.: F-regularity, test elements, and smooth base change. Trans. Am. Math. Soc. **346**, 1–62 (1994)

[HH4] Hochster, M. and Huneke, C.: Tight closure of parameter ideals and splitting in module-finite extensions. J. Algebr. Geom. **3**, 599–670 (1994)
[Li] Lipman, J.: Rational singularities, with applications to algebraic surfaces and unique factorization. Publ. Math., Inst. Hautes Étud. Sci. **36**, 195–279 (1969)
[Ma] Ma, F.: Splitting in integral extensions, Cohen-Macaulay modules and algebras. J. Algebra **116**, 176–195 (1988)
[Mc] McDermott, M.: *Tight closure, plus closure and Frobenius closure in cubical cones*. Thesis, University of Michigan, 1996
[Mu] Muir, T.: *A treatise on the theory of determinants, Revised and enlarged by William H. Metzler*. Longmans, Green and co., New York, London, 1933
[Ro] Roberts, P.: A computation of local cohomology. Contemp. Math. **159**, 351–356 (1994)
[Si1] Singh, A.K.: Failure of F-purity and F-regularity in certain rings of invariants. Ill. J. Math. **42**, 441–448 (1998)
[Si2] Singh, A.K.: A computation of tight closure in diagonal hypersurfaces. J. Algebra **203**, 579–589 (1998)
[Si3] Singh, A.K.: $\mathbb{Q}$-Gorenstein splinter rings of characteristic p are F-regular. Math. Proc. Camb. Philos. Soc., to appear
[Sm] Smith, K.E.: Tight Closure of parameter ideals. Invent. Math. **115** 41–60 (1994)