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A note on Bockstein homomorphisms in local cohomology

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Abstract.

This note is an exposition on the results from [SiW] related to Lyubeznik's conjecture on local cohomology. We use the method of Bockstein homomorphisms, which to our knowledge has not employed before in commutative algebra. As an example, we strengthen the classical Stanley–Reisner correspondence by relating Bockstein homomorphisms on local cohomology in monomial rings to the topological Bockstein homomorphisms of simplicial complexes.

We introduce the necessary language in Section 1, then present the motivation with our main theorem in Section 2, and finally illustrate our results in the case of Stanley–Reisner ideals.

§1. Basic notions

Definition 1.1. Let R be a commutative Noetherian ring, and M an R-module. Let p be an element of R which is a non-zero divisor on M. Let $\mathrm{F}^\bullet$ be a covariant δ-functor on the category of R-modules. The exact sequence

$$0 \longrightarrow M \xrightarrow{\ p\ } M \longrightarrow M/pM \longrightarrow 0$$

then induces an exact sequence

$$\mathrm{F}^k(M/pM) \xrightarrow{\ \delta_p^k\ } \mathrm{F}^{k+1}(M) \xrightarrow{\ \mathrm{F}^{k+1}(p)\ } \mathrm{F}^{k+1}(M) \xrightarrow{\ \pi_p^{k+1}\ } \mathrm{F}^{k+1}(M/pM)\,.$$

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The composition

$$\pi_p^{k+1} \circ \delta_p^k \colon \mathrm{F}^k(M/pM) \to \mathrm{F}^{k+1}(M/pM)$$

is the *Bockstein homomorphism* $\beta_p^k = \beta_p^k(F(M))$.

An elementary check shows that β_p^k is the connecting homomorphism that results from applying $\mathrm{F}^\bullet$ to the short exact sequence

$$0 \longrightarrow M/pM \xrightarrow{\ p\ } M/p^2M \longrightarrow M/pM \longrightarrow 0.$$

Let $\mathfrak{a}$ be an ideal of R, generated by elements $f_1, \dots, f_t$. The covariant δ-functors of interest to us are local cohomology $H^\bullet_{\mathfrak{a}}(-)$, Koszul cohomology $H^\bullet(f_1, \dots, f_t; -)$, and $\mathrm{Ext}^\bullet_R(R/\mathfrak{a}, -)$ discussed briefly below.

Setting $\boldsymbol{f}^e = f_1^e, \dots, f_t^e$, the Koszul cohomology modules $H^\bullet(\boldsymbol{f}^e; M)$ are the cohomology modules of the Koszul complex $K^\bullet(\boldsymbol{f}^e; M)$. For each $e \geq 1$, one has a map of complexes

$$K^\bullet(\boldsymbol{f}^e; M) \to K^\bullet(\boldsymbol{f}^{e+1}; M)\,,$$

and thus a filtered directed system $\{K^\bullet(\boldsymbol{f}^e; M)\}_{e \geq 1}$. The direct limit of this system can be identified with the Čech complex $\check{C}^\bullet(\boldsymbol{f}; M)$ given by

$$0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{i<j} R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_t} \to 0\,.$$

The local cohomology modules $H^\bullet_{\mathfrak{a}}(M)$ may be computed as the cohomology modules of $\check{C}^\bullet(\boldsymbol{f}; M)$, or equivalently as the direct limit of the Koszul cohomology modules $H^\bullet(\boldsymbol{f}^e; M)$: the filter condition yields an isomorphism of functors

$$H^k_{\mathfrak{a}}(-) \;\cong\; \varinjlim_e H^k(\boldsymbol{f}^e; -)\,,$$

and each element of $H^k_{\mathfrak{a}}(M)$ lifts to an element of $H^k(\boldsymbol{f}^e; M)$ for $e \gg 0$.

Remark 1.2. Let $\mathrm{F}^\bullet$ and $G^\bullet$ be covariant δ-functors on the category of R-modules, and let $\tau \colon \mathrm{F}^\bullet \to G^\bullet$ be a natural transformation. The Bockstein homomorphism is functorial:one has a commutative diagram

$$\begin{array}{ccc} \mathrm{F}^k(M/pM) & \longrightarrow & \mathrm{F}^{k+1}(M/pM) \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ G^k(M/pM) & \longrightarrow & G^{k+1}(M/pM)\,, \end{array}$$

where the horizontal maps are the respective Bockstein homomorphisms. With $\mathfrak{a} = (f_1, \dots, f_k)$, the natural transformations of interest to us are

$$\operatorname{Ext}^\bullet(R/\mathfrak{a}^e, -) \to H^\bullet_{\mathfrak{a}}(-)\,,$$

$$H^\bullet(\boldsymbol{f}^e; -) \to H^\bullet(\boldsymbol{f}^{e+1}; -)\,,$$

and

$$H^\bullet(\boldsymbol{f}^e; -) \to H^\bullet_{\mathfrak{a}}(-)\,.$$

Let M be an R-module, let p be an element of R which is a non-zero divisor on M, and let $\boldsymbol{f} = f_1, \dots, f_t$ and $\boldsymbol{g} = g_1, \dots, g_t$ be elements of R such that $f_i \equiv g_i \mod p$ for each i. One then has isomorphisms

$$H^\bullet(\boldsymbol{f}; M/pM) \cong H^\bullet(\boldsymbol{g}; M/pM)\,.$$

Example 1.3. Let p be a non-zero divisor on R. Let x be an element of R. The Bockstein homomorphism on Koszul cohomology $H^\bullet(x; R/pR)$ is

$$\begin{array}{rcl} (0 :_{R/pR} x) = H^0(x; R/pR) & \to & H^1(x; R/pR) = R/(p,x)R \\ r \mod (p) & \mapsto & rx/p \mod (p,x)\,. \end{array}$$

Let y be an element of R with $x \equiv y \mod p$. Comparing the Bockstein homomorphisms β, β' on $H^\bullet(x; R/pR)$ and $H^\bullet(y; R/pR)$ respectively, one sees that the diagram

$$\begin{array}{ccc} \big(0 :_{R/pR} x\big) & \xrightarrow{\ \beta\ } & R/(p,x)R \\ \| & & \| \\ \big(0 :_{R/pR} y\big) & \xrightarrow{\ \beta'\ } & R/(p,y)R \end{array}$$

commutes if and only if $(x-y)/p \in (p,y)R$.

A key point in the proof of Theorem 2.2 is Lemma 1.4, which states that upon passing to the direct limits $\varinjlim_e H^\bullet(\boldsymbol{f}^e; M/pM)$ and $\varinjlim_e H^\bullet(\boldsymbol{g}^e; M/pM)$, the phenomenon of Example 1.3 disappears: the Bockstein homomorphisms always commute with the induced isomorphisms

$$\varinjlim_e H^\bullet(\boldsymbol{f}^e; M/pM) \cong \varinjlim_e H^\bullet(\boldsymbol{g}^e; M/pM)\,.$$

On the other hand, one has the following lemma.

Lemma 1.4 ([SiW]). *Let M be an R-module, and let p be an element of R which is M-regular. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R with $\sqrt{(\mathfrak{a}+pR)} = \sqrt{(\mathfrak{b}+pR)}$. Then there exists a commutative diagram*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^k_{\mathfrak{a}}(M/pM) & \longrightarrow & H^{k+1}_{\mathfrak{a}}(M/pM) & \longrightarrow & \cdots \\ & & \big\downarrow & & \big\downarrow & & \\ \cdots & \longrightarrow & H^k_{\mathfrak{b}}(M/pM) & \longrightarrow & H^{k+1}_{\mathfrak{b}}(M/pM) & \longrightarrow & \cdots \end{array}$$

where the horizontal maps are the respective Bockstein homomorphisms, and the vertical maps are isomorphisms.

§2. Motivation and main result

Huneke [H, Problem 4] asked whether local cohomology $H^{\bullet}_{\mathfrak{a}}(R)$ modules of Noetherian rings R have finitely many associated prime ideals.

Suppose p is a prime number such that the Bockstein homomorphism

$$\beta^k_p \colon H^k_{\mathfrak{a}}(R/pR) \to H^{k+1}_{\mathfrak{a}}(R/pR)$$

is nonzero. Clearly, $H^{k+1}_{\mathfrak{a}}(R)$ must then have an associated prime containing p. But different prime numbers cannot be in the same associated prime ideal, so Huneke's conjecture implies the vanishing of all but finitely many β^k_p with p a prime number.

Example 2.1. Consider the hypersurface

$$R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)$$

and ideal $\mathfrak{a} = (x, y, z)R$. A variation of the argument given in [Si2] shows that

$$\beta^2_p \colon H^2_{\mathfrak{a}}(R/pR) \to H^3_{\mathfrak{a}}(R/pR)$$

is nonzero for each prime integer p. Hence, $H^3_{\mathfrak{a}}(R)$ has infinitely many associated primes

Lyubeznik conjectured in [L1, Remark 3.7] that for regular rings R, each local cohomology module $H^k_{\mathfrak{a}}(R)$ has finitely many associated prime ideals. This conjecture has been verified for regular rings of positive characteristic by Huneke and Sharp [HS], and for regular local rings of characteristic zero as well as unramified regular local rings of mixed characteristic by Lyubeznik [L1, L2]. Surprisingly, it remains unresolved for polynomial rings over $\mathbb{Z}$ where the issue of p-torsion appears to be central in studying Lyubeznik's conjecture. The following theorem provides supporting evidence for Lyubeznik's conjecture.

Theorem 2.2 ([SiW])**.** *Let $R = \mathbb{Z}[x_1, \dots, x_n]$ be a polynomial ring in finitely many variables over the ring of integers. Let $\mathfrak{a} = (f_1, \dots, f_t)$ be an ideal of R, and let p be a prime integer.*

If p is not a zero divisor on the Koszul cohomology module $H^{k+1}(\boldsymbol{f}; R)$, then the Bockstein homomorphism $\beta_p^k \colon H_{\mathfrak{a}}^k(R/pR) \to H_{\mathfrak{a}}^{k+1}(R/pR)$ is zero.

We outline the proof of Theorem 2.2. One investigates the endomorphism φ of R with $\varphi(x_i) = x_i^p$ for each i. Since

$$H^{k+1}(\boldsymbol{f}; R) \xrightarrow{\ p\ } H^{k+1}(\boldsymbol{f}; R)$$

is injective and φ is flat, it follows that

$$H^{k+1}(\varphi^e(\boldsymbol{f}); R) \xrightarrow{\ p\ } H^{k+1}(\varphi^e(\boldsymbol{f}); R)$$

is injective for each $e \geq 0$. Thus, the Bockstein map on Koszul cohomology

$$H^k(\varphi^e(\boldsymbol{f}); R/pR) \longrightarrow H^{k+1}(\varphi^e(\boldsymbol{f}); R/pR)$$

must be the zero map. While $\{\varphi^e(\boldsymbol{f})R\}$ and $\{\boldsymbol{f}^e R\}$ are not cofinal families of ideals, Lemma 1.4 allows to replace one by the other for the purpose of studying Bockstein homomorphisms on local cohomology.

Suppose $\eta \in H_{\mathfrak{a}}^k(R/pR)$. Then η has a lift in $H^k(\varphi^e(\boldsymbol{f}); R/pR)$ for large e, and the commutativity of the diagram

$$\begin{array}{ccc}
H^k(\varphi^e(\boldsymbol{f}); R/pR) & \longrightarrow & H^{k+1}(\varphi^e(\boldsymbol{f}); R/pR) \\
\downarrow & & \downarrow \\
H^k_{(\varphi^e(\boldsymbol{f}))}(R/pR) & \longrightarrow & H^{k+1}_{(\varphi^e(\boldsymbol{f}))}(R/pR) \\
\downarrow & & \downarrow \\
H^k_{(\boldsymbol{f})}(R/pR) & \longrightarrow & H^{k+1}_{(\boldsymbol{f})}(R/pR),
\end{array}$$

(each horizontal map is a Bockstein homomorphism) implies that η maps to zero in $H_{\mathfrak{a}}^{k+1}(R/pR)$.

§3. Stanley–Reisner theory

Definition 3.1. Let Δ_n be the n-simplex, viewed as a simplicial complex (i.e., with its collection of faces), and fix a subcomplex Δ. Then the associated *Stanley–Reisner ideal* is the ideal

$$\mathfrak{a}_\Delta = (\boldsymbol{x}^\sigma \mid \sigma \notin \Delta) \subseteq R$$

generated by the monomials $\boldsymbol{x}^\sigma = \prod_{i=0}^n x_i^{\sigma_i}$ for which σ is not a face of Δ. The quotient $S_\Delta = R/\mathfrak{a}_\Delta$ is the *Stanley–Reisner ring* of Δ.

For $\boldsymbol{a} \in \mathbb{Z}^{n+1}$ write $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$ for the sets $\{i \mid a_i > 0\}$ and $\{i \mid a_i < 0\}$ respectively. Both can be viewed as subsimplices of Δ_n. Hochster's theorem, [BH, Section 5.3] states the following, with notation explained below:

$$\left(H^i_{\mathfrak{m}}(S_\Delta/pS_\Delta)\right)_{\boldsymbol{a}} = \tilde{H}^{i-|\boldsymbol{a}_-|-1}(\operatorname{link}_{\star_\Delta(\boldsymbol{a}_+)}(\boldsymbol{a}_-); \mathbb{Z}/p\mathbb{Z}).$$

Definition 3.2. Let Δ be a simplicial complex and suppose τ is a subset of its vertex set (but not necessarily a simplex). The *link*, $\operatorname{link}_\Delta(\tau)$, is the set

$$\operatorname{link}_\Delta(\tau) = \{\tau' \mid \tau \cap \tau' = \emptyset, \quad \tau \cup \tau' \in \Delta\},$$

while its *star*, $\star(\tau)$, is simply

$$\star_\Delta(\tau) = \{\tau' \mid \tau \cup \tau' \in \Delta\}.$$

The Bockstein construction respects the identifications of Hochster's result:

Theorem 3.3 ([SiW])**.** *Let $\Delta \subseteq \Delta_n$ be a simplicial complex on $n+1$ vertices, and let $S_\Delta = R/\mathfrak{a}_\Delta$ be its integral Stanley–Reisner ring. If p is a prime number then the following are equivalent:*

(1) *For some multi-degree $\boldsymbol{a}$, the topological Bockstein morphism on the singular reduced $(i - |\boldsymbol{a}_-| - 1)$-th $\mathbb{Z}/p\mathbb{Z}$-cohomology of $\operatorname{link}_{\star_\Delta(\boldsymbol{a}_+)}(\boldsymbol{a}_-)$ is nonzero.*

(2) *The Bockstein morphism $\beta_p^{n+1-i}(H_{\mathfrak{a}_\Delta}(R))$ on the local cohomology group $H^{n+1-i}_{\mathfrak{a}_\Delta}(R/pR)$ is nonzero.*

An ingredient for this is that Bocksteins commute with local duality:

Theorem 3.4 ([SiW])**.** *Let $(S, \mathfrak{m}, \mathbb{K})$ be a local Gorenstein ring of dimension at least one. Let $p \in S$ be a regular element on both ω_S and on the finitely generated S-module M.*

The Bockstein morphism on $\operatorname{Ext}_S^{\dim(S)-i}(M, \omega_S/p\omega_S)$ induced by

$$0 \to \omega_S \xrightarrow{p} \omega_S \to \omega_{S/pS} = \omega_S/p\omega_S \to 0$$

is Matlis dual to the Bockstein morphism on $H^i_{\mathfrak{m}}(M/pM)$ induced by

$$0 \to M \xrightarrow{p} M \to M/pM \to 0.$$

Example 3.5. Let Λ_m be the *m-fold dunce cap*, i.e., the quotient of the unit disk obtained by identifying each point on the boundary circle with its translates under rotation by $2\pi/m$; the 2-fold dunce cap Λ_2 is the real projective plane.

Suppose m is the product of distinct primes $p_1, \dots, p_r$. It is readily computed that the Bockstein homomorphisms

$$\widetilde{H}^1(\Lambda_m; \mathbb{Z}/p_i) \to \widetilde{H}^2(\Lambda_m; \mathbb{Z}/p_i)$$

are nonzero. Let Δ be the simplicial complex corresponding to a triangulation of Λ_m, and let $\mathfrak{a}$ in $R = \mathbb{Z}[x_1, \dots, x_n]$ be the corresponding Stanley–Reisner ideal. The theorem then implies that the Bockstein homomorphisms

$$H^{n-3}_{\mathfrak{a}}(R/p_iR) \to H^{n-2}_{\mathfrak{a}}(R/p_iR)$$

are nonzero for each p_i. It follows that the local cohomology module $H^{n-2}_{\mathfrak{a}}(R)$ has a p_i-torsion element for each $i = 1, \dots, r$.

If $m = 2$ (with standard triangulation shown below), Δ has singular cohomology

$$H^i(\Delta; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0; \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i = 1, 2; \\ 0 & \text{else.} \end{cases}$$

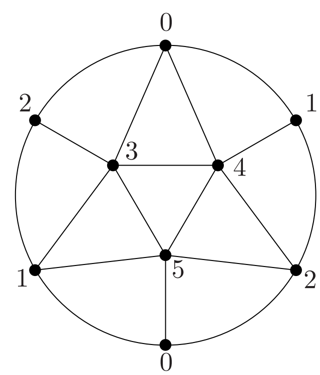

Fig. 1. The minimal triangulation of the real projective plane

The associated Stanley–Reisner ideal is

$$\begin{aligned} \mathfrak{a}_{\mathbb{RP}^2} &= (x_1x_2x_5,\, x_1x_2x_0,\, x_1x_3x_4,\, x_1x_3x_0,\, x_1x_4x_5, \\ &\quad x_2x_3x_4,\, x_2x_3x_5,\, x_2x_4x_0,\, x_3x_5x_0,\, x_4x_5x_0) \\ &\quad \subseteq \mathbb{Z}[x_0,\dots,x_5]; \end{aligned} \tag{1}$$

the minimal free resolution is of the form

$$R \to R^7 \to R^{15} \to R^{10} \to R$$

where the top degree map is the transpose of the the peculiar matrix

$$R^7 \xrightarrow{(x_1,x_2,x_3,x_4,x_5,x_6,2)} R.$$

One may show that the last entry in this map carries topological meaning and is "responsible" for β_p^2 to be nonzero on $H^3_{\mathfrak{a}}(R/pR)$.

Remark 3.6. For a δ-functor F, vanishing of all p-Bocksteins on M is certain if $\mathrm{F}^i(M)$ is torsionfree for all i. However, it is possible that (for fixed p) the p-Bockstein homomorphisms are all zero and yet $\mathrm{F}^i(M)$ has p-torsion. For example, the 4-fold dunce cap Λ_4 has non-vanishing p-Bocksteins ($p \in \mathbb{N}_+$) if and only if $p = 4u$, u odd. (In this case, $H^1(\Lambda_4;\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ while $H^2(\Lambda_4;\mathbb{Z}) = 0$. So $H^1(\Lambda_4,\mathbb{Z}/p\mathbb{Z}) = H^2(\Lambda_4;\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/\gcd(4,p)\mathbb{Z}$. The p-Bockstein homomorphism linking these two, when interpreted as endomorphism of $\mathbb{Z}/\gcd(4,p)\mathbb{Z}$ is multiplication by $4/\gcd(4,p)$.)

In general, there is a spectral sequence relating Bockstein homomorphisms to varying powers of p. It should be noted that it is possible that the Bockstein homomorphisms to all powers of p vanish and yet $\mathrm{F}(M)$ has p-torsion. For example, $R = \mathbb{Z}$, p a prime number, $\mathrm{F}^i = H^i_p(-)$ has for $M = R$ only one nonzero $\mathrm{F}^i(M/pM)$, namely when $i = 0$. So nonzero Bocksteins cannot exist but clearly $\mathrm{F}^1(M)$ is nonzero and p-torsion.

References

[BH] W. Bruns and J. Herzog, Cohen–Macaulay rings, revised ed., Cambridge Stud. Adv. Math., **39**, Cambridge Univ. Press, Cambridge, 1998.

[H] C. Huneke, Problems on local cohomology, In: Free Resolutions in Commutative Algebra and Algebraic Geometry, Sundance, Utah, 1990, Res. Notes Math., **2**, Jones and Bartlett, Boston, MA, 1992, pp. 93–108.

[HS] C. Huneke and R. Sharp, Bass numbers of local cohomology modules, Trans. Amer. Math. Soc., **339** (1993), 765–779.

[L1] G. Lyubeznik, Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra), Invent. Math., **113** (1993), 41–55.

[L2] G. Lyubeznik, Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case, Comm. Alg., **28** (2000), 5867–5882.

[Si2] A. K. Singh, p-torsion elements in local cohomology modules, Math. Res. Lett., **7** (2000), 165–176.

[SiW] A. K. Singh and U. Walther, Bockstein homomorphisms in local cohomology, Crelle's J. Reine Angew. Math., to appear.

[24h] S. B. Iyengar, G. J. Leuschke, A. Leykin, C. Miller, E. Miller, A. K. Singh and U. Walther, Twenty-Four Hours of Local Cohomology, Grad. Stud. Math., **87**, Amer. Math. Soc., Providence, RI, 2007, xviii+282 pp.

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