

Final Exam Practice Problems

Note: This is a (not necessarily exhaustive) list of the concepts you will need to know for the upcoming final exam. All of the homework problems will be useful in studying for the exam as well. If you have any questions, please do not hesitate to ask.

1. All of the problems in the practice problem set that is also posted online.
2. Consider the following system of equations:

$$\begin{cases} x + 2y + 4z = 1 \\ x + y + z = 0 \\ -x + z = 0 \end{cases}$$

- (a) Write this system as a matrix equation.
 - (b) Use Gaussian elimination to solve this system.
3. Suppose you are given 2 unit vectors in \mathbb{R}^2 , call them \mathbf{u} and \mathbf{v} . Can $\mathbf{u} \cdot \mathbf{v} = 2$? Why or why not?
 4. Suppose A is an $n \times n$ matrix with real entries.
 - (a) Define what it means for A to be invertible.
 - (b) Suppose A is invertible. Prove that the inverse of A is unique.
 - (c) If A is invertible, what is the only solution to $A\mathbf{x} = \mathbf{b}$? Why is this the only solution?
 - (d) If

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix},$$

what is A^{-1} ? Please show your work.

5. Suppose R is an $m \times n$ matrix. Prove that $R^T R$ is symmetric.
6. Suppose you are given a system of linear equations with 2 equations and 2 unknowns. We know that there can be no solution, infinitely many solutions, or one solution. Draw a possible row picture for each of these scenarios (there is more than one correct answer).
7. Suppose V is a vector space.
 - (a) What does it mean for V to be closed under addition?
 - (b) Is the set of all 2×2 invertible matrices closed under addition? If so, please prove it. If not, please provide a counter-example.

8. Let V be the space of all 3×3 matrices with real entries. Is W , the set of all 3×3 lower triangular matrices, a subspace of V ? Why or why not?
9. What are 3 ways of determining whether or not a given $n \times n$ matrix A is invertible?
10. Suppose A is an $m \times n$ matrix, with the form

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$$

where \mathbf{a}_i are column vectors in \mathbb{R}^m ($i = 1, 2, \dots, n$).

- (a) Give the definition of the column space of A , denoted $C(A)$.
- (b) Prove that $C(A)$ is a subspace of \mathbb{R}^m . (Hint: If \mathbf{x} is a vector in $C(A)$, how can you express \mathbf{x} in terms of the columns of A ?)
11. Know how to prove whether or not a given space W is a subspace of a vector space. (Section 3.1–10,17).
12. Know how to find the complete solution to $A\mathbf{x} = \mathbf{b}$. (Recall: $\mathbf{x} = \mathbf{x}_n + \mathbf{x}_p$, where \mathbf{x}_n is in the nullspace of A and \mathbf{x}_p is a particular solution to $A\mathbf{x} = \mathbf{b}$). (Section 3.4–1,3,4,6,33).
13. Know how to prove that a set of vectors is linearly independent or dependent. Also, know how to show that a given set of vectors spans a vector space. Also, know the definition of a basis and how to apply this definition to a given set of vectors to determine whether or not they form a basis for a given vector space. See page 200 especially! (Section 3.5–2,5,7,8,11,15)
14. What are the orthogonality relations between the Four Fundamental Subspaces? In particular, know the Fundamental Theorem of Linear Algebra, Part II (page 198). (Section 4.1–4,5).
15. Know how to find the projection of a vector \mathbf{x} in \mathbb{R}^n onto a subspace of \mathbb{R}^n . The key is not to memorize the formulas—understand where they come from. (Section 4.2–2,10,17,24,27).
16. What is the definition of an orthogonal matrix? Orthogonal matrices are nice because $Q^T = ?$ Also, know how to show that for any \mathbf{x} in \mathbb{R}^n , $\|Q\mathbf{x}\| = \|\mathbf{x}\|$, where Q is an $n \times n$ orthogonal matrix. Also, remember that $Q^T Q = I$ for any matrix with orthonormal columns, not just square matrices. If the matrix happens to be square, then it is called *orthogonal*.
17. Know how to compute the determinant of a matrix.
18. What are the eigenvalues and eigenvectors of a projection matrix? (Consider the case in which the projection matrix P projects vectors from \mathbb{R}^3 onto a plane R spanned by the vectors \mathbf{a}_1 and \mathbf{a}_2 , for example).
19. Know how to find eigenvalues and eigenvectors of a 2×2 matrix. Also, why does $\det(A - \lambda I) = 0$ give the eigenvalues of A ?

20. Know the Spectral Theorem (don't worry about the proof, though) and how to apply it to a given matrix.
21. Know how to diagonalize a matrix (Problem 16 on page 309 would be good practice).
22. What is the definition of a symmetric positive definite matrix? Make sure you understand why test number 5 works in the blue box on page 344.
23. Know how to determine whether or not a given transformation is linear or not.
24. Know how to find the matrix of a linear transformation.