Maple Lab 2
Due Friday, August 5, by 10 a.m.

Instructions This Maple lab is intended to introduce you to the singular value decomposition of a matrix (which always exists) and to explore a simple model about glucose-insulin interactions in the human digestive system. There are some questions that you will be required to answer–these are in italics and indicated by letters (a), (b), etc. Please answer these using Maple and/or on a separate sheet of paper. If you use a separate piece of paper, please indicate if your answers are on the Maple worksheet. If you have any questions, please do not hesitate to ask. Finally, there are a couple of sample worksheets posted on the class website that should be helpful.

1. This first problem is concerned with the *singular value decomposition* (svd) of the matrix A. The svd of a matrix reveals a lot about how the matrix acts on vectors. If A is an $m \times n$ matrix with real entries, the svd of A is

$$A = U\Sigma V^T,$$

where U is an orthogonal $m \times m$ matrix, V is an orthogonal $n \times n$ matrix, and Σ (the capital Greek letter sigma) is an $m \times n$ diagonal matrix (it is also rectangular–we will justify what we mean by a diagonal rectangular matrix momentarily). Note that the above formula can be rewritten by multiplying both sides by V to obtain

$$AV = U\Sigma.$$

The columns of U, denoted by $\mathbf{u}_1, \ldots, \mathbf{u}_m$, are called the *left singular vectors* of A; the columns of V, denoted by $\mathbf{v}_1, \ldots, \mathbf{v}_n$, are called the *right singular vectors* of A; the diagonal entries of Σ, denoted by $\sigma_1, \ldots, \sigma_p$, are called the *singular values* of A. (We use p to denote the minimum of m and n). We also order the singular values such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$. Then, in the case where $m > n$ (so $p = n$) the svd of A looks something like:

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_p \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

In order to simplify the notation, we'll just consider the case where $m = 2$ and $n = 2$, so the svd takes the form

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \mathbf{u}_1 & \mathbf{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \tag{1}$$

First of all, we note that the singular values are all nonnegative. That is, some of the singular values may be zero. In fact, the rank of the matrix A is equal to the number of nonzero

singular values (this is one method MATLAB could to determine the rank of a matrix–find the singular values and count the number of nonzero singular values). Thus, if $\sigma_1 = 2$ but $\sigma_2 = 0$, then the rank of A is one. If σ_1 and σ_2 are both positive, then the rank of A is two. If we multiply the matrices in (1), we obtain

$$\begin{bmatrix} | & | \\ A\mathbf{v}_1 & A\mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 \\ | & | \end{bmatrix} \tag{2}$$

Thus $A\mathbf{v}_1 = \sigma_1\mathbf{u}_1$ and $A\mathbf{v}_2 = \sigma_2\mathbf{u}_2$. Those last two equations have a lot of meaning. Notice, first of all, that they are *almost* like the equation for eigenvalues. The difference is that, in general, $\mathbf{v}_1$ and $\mathbf{u}_1$ may live in different vector spaces (that is, in general, $\mathbf{v}_1$ lives in $\mathbb{R}^n$ but $\mathbf{u}_1$ lives in $\mathbb{R}^m$, where n and m may be different; here we are considering $m = n = 2$ to simplify the matter).

(a) *Please fill in the blanks in the following paragraph.*

As an example, consider the case where $\sigma_1 > 0$ but $\sigma_2 = 0$, so the rank of A is 1.

i. Then we have $A\mathbf{v}_1 = \sigma_1\mathbf{u}_1$, so $\mathbf{u}_1$ lives in the _____ space of A. The dimension of this space is ___. Thus $\mathbf{u}_1$ forms a basis for this fundamental subspace of A.

ii. Also, $A\mathbf{v}_2 = \sigma_2\mathbf{u}_2 = \mathbf{0}$, so $\mathbf{v}_2$ lives in the _____ space of A. The dimension of this space is ___. Thus $\mathbf{v}_2$ forms a basis for this fundamental subspace of A.

iii. We created the svd in such a way that $\mathbf{u}_2$ is orthogonal to $\mathbf{u}_1$. This means that $\mathbf{u}_2$ is in the _____ space of A. Since the dimension of this fundamental subspace is ___, $\mathbf{u}_2$ forms a basis for this fundamental subspace of A.

iv. Finally, we know that $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$. Thus $\mathbf{v}_1$ must be in the _____ space of A. Since the dimension of this fundamental subspace is ___, $\mathbf{v}_1$ forms a basis for this fundamental subspace of A.

The point of the above exercise is to show one of the (many) powerful applications of the svd. It contains orthonormal bases for all four fundamental subspaces of A! It also gives us all of the dimensions of the four fundamental subspaces–since the number of nonzero singular values gives us the rank of A (dimension of the column space of A), we can use the Fundamental Theorem of Linear Algebra to deduce the dimensions of the remaining 3 fundamental subspaces.

We can also use the svd to determine geometric properties of the matrix A. This is where Maple comes in handy. Our goal is to determine what happens to vectors that live in the unit circle when we apply the matrix A, where

$$A = \begin{bmatrix} 1351/780 & 2 \\ 0 & 1351/780 \end{bmatrix}.$$

First, recall that any vector on the unit circle can be represented as

$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$

We want to have a large collection of these vectors, so we can get a good idea of what the matrix A does to the unit circle.

(b) *Use the* seq *command in Maple (see the sample online) to create a sequence of vectors on the unit circle.*

Your result will be something of the form

$$C = \begin{bmatrix} \cos(\theta_1) & \cos(\theta_2) & \dots & \cos(\theta_{200}) \\ \sin(\theta_1) & \sin(\theta_2) & \dots & \sin(\theta_{200}) \end{bmatrix}$$

Each column of this matrix is a different vector on the unit circle.

(c) *Plot this and the marker points to see the unit circle (see the commands in the online sample).*

Now we want to determine what happens to the unit circle when we apply V^T. Since V is an orthogonal matrix, so is V^T (check this–it is true for *any* orthogonal matrix), so $\|V^T\mathbf{x}\| = \|\mathbf{x}\|$ for any vector $\mathbf{x}$ in $\mathbb{R}^2$.

(d) *In Maple, compute the matrix product $V^T C$.*

This is the same as applying V^T to each column of C separately and storing the results in a new matrix–the first row contains the new x-coordinates and the second row contains the new y-coordinates of the transformed vectors.

(e) *Plot the results (see the sample online for one way to do this), and explain what you see–what happened to the unit circle? The markers will be helpful.*

(f) *Now, multiply the last result by Σ and plot the results–you should see an ellipse. How are the lengths of the semi-axes related to the singular values of A?*

(g) *Finally, multiply this last result by U and plot the results. What happened to the ellipse?*

In general, any 2×2 matrix transforms the unit circle into an ellipse (if it's invertible). If the original matrix is not invertible, $\sigma_2 = 0$. Thus the length of one of the semiaxes of the ellipse will be zero, so it will be a line. Check this by changing the original matrix A to

$$\begin{bmatrix} 93/65 & 88/65 \\ 93/65 & 88/65 \end{bmatrix}.$$

Don't worry about printing the results in this case–it is just for you to see what happens if A is not invertible.

Summary: The svd stores a wealth of information about the matrix A. However, the svd is much more than a wonderful theoretical tool. For example, the svd can be used to compress images. It is also used in many computer algorithms.

2. This problem is about a simplified model of the interactions between insulin and glucose in the human digestive system. After a meal, the amount of excess glucose in the body increases (where the excess glucose is the amount above the fasting level–it can be measured in milligrams per milliliter, for example). This, in turn, stimulates the $\beta-$cells of the pancreas to release the hormone insulin. Insulin "tells" certain cells in the body to absorb the glucose. Type I diabetes is an autoimmune disease in which the immune system attacks the $\beta-$cells. With no $\beta-$cells, the body cannot produce insulin. The cause of the body's attack on the $\beta-$cells is still unknown, and a cure for Type I diabetes has not yet been found. However, people with Type I diabetes can give themselves insulin to help remedy the problem.

Suppose one eats a meal at time $k = 0$ (in this model we will use k to represent the number of hours after the meal). The amount of excess glucose at time k will be denoted by $g(k)$ and

the amount of excess insulin k hours after the meal will be denoted by $h(k)$. A simplified model is given by

$$\begin{cases} g(k+1) = ag(k) - bh(k) \\ h(k+1) = cg(k) + dh(k). \end{cases} \tag{3}$$

The coefficients a, b, c, and d are positive constants. In particular, $g(k+1)$ is the amount of excess glucose at time $k+1$ (that is, one hour later), and $h(k+1)$ is the amount of excess insulin at time $k+1$. The term $-bh(k)$ appears because insulin causes the amount of excess glucose to decrease. The term $cg(k)$ is in the second equation because the presence of glucose stimulates the release of insulin by the pancreatic $\beta-$cells.

Let's insert $g(k)$ and $h(k)$ into a vector, say $\mathbf{u}_k$, so

$$\mathbf{u}_k = \begin{bmatrix} g(k) \\ h(k) \end{bmatrix} \tag{4}$$

Then $\mathbf{u}_0 = (g(0), h(0))$ is the initial amount of excess glucose and excess insulin present.

(a) *Notice that we can rewrite* (3) *as* $\mathbf{u}_{k+1} = A\mathbf{u}_k$. *What is the matrix A?*

It can be shown that the solution to this type of equation is $\mathbf{u}_k = A^k \mathbf{u}_0$. Since matrix powers are not easy to compute, we turn to eigenvalues for help. We will assume A is diagonalizable, so there is an invertible matrix S (containing eigenvectors of A) and a diagonal matrix Λ (containing eigenvalues of A) such that

$$A = S\Lambda S^{-1}.$$

Then $A^k = S\Lambda^k S^{-1}$, and so $\mathbf{u}_k = S\Lambda^k S^{-1} \mathbf{u}_0$.

For our example, we'll use the parameters $a = 0.9$, $b = 0.4$, $c = 0.1$, and $d = 0.9$. Also, let's suppose that right after a meal, we have $\mathbf{u}_0 = (100, 0)$.

(b) *Use Maple to compute the eigenvectors and eigenvalues of A, and create the matrices S and Λ (see the sample worksheet).*

Notice that Maple uses I to denote the imaginary number $\sqrt{-1}$. In the sample worksheet, there is a function $V(k)$ that is created using the *unapply* command. It creates the matrix $S\Lambda^k S^{-1}$, where k is the independent variable. If you type $V(7)$, the output will be the matrix $S\Lambda^7 S^{-1}$. We can plot many points to see what happens over a long period of time. The commands for doing this are all given in the sample worksheet online. The idea is to compute

$$\mathbf{u}_k = \begin{bmatrix} g(k) \\ h(k) \end{bmatrix} = S\Lambda^k S^{-1} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

for many different values of k, and then plot these points (Maple will also connect them with lines). In the first figure that you plot, $g(k)$ is on the $x-$axis and $h(k)$ is on the $y-$axis. This spiral shape is characteristic of complex eigenvalues. It may be easier to understand what is going on if we plot k on the $x-$axis and $g(k)$ and $h(k)$ on the $y-$axis. Suggestions for how to do this are all given in the sample worksheet. Notice the oscillating behavior (this is also described by the spiral picture you plotted first). That is, right after the meal is consumed and the amount of excess glucose is high, insulin is produced, which lowers the amount of glucose to below the fasting level. When there is

so little glucose present, the insulin level also drops. Notice that it takes approximately 7 hours (with our parameters) for the glucose to drop below fasting level. Also, at 7 hours, there is a lot of extra insulin present, which causes the glucose level to drop below the fasting level. The glucose and insulin levels continue to oscillate. This is not a perfect model by any means (for example, where does the extra glucose come from at 29 hours?), but it provides us with a basic understanding of how glucose and insulin interact, especially for the first few hours. Change c to 0.01 to see what might happen for someone with diabetes (this means that the presence of glucose does not have much of an effect on the production of insulin)–the glucose level drops, but very slowly, and the insulin peaks near 4 instead of 30.

To correct for this, a person with Type I diabetes must give himself or herself insulin. We can include this in our model as well. For example, with $c = 0.01$, set $\mathbf{u}_0 = (100, 30)$.

(c) *Please produce a plot with g on the $x-$axis and h on the $y-$axis, as well as plots with k on the $x-$axis and g and h on the $y-$axis.*

Now the person with diabetes is able to absorb all of the excess glucose in approximately the same amount of time as a person without diabetes.

Summary: This problem illustrates one of the important applications of eigenvalues and eigenvectors. Equation (3) is called a *difference equation.* If we move to the case where our time values are continuous instead of discrete, we obtain *differential equations.* If one has a set of differential equations that is linear (and where the coefficients a, b, c, d, etc. are constant), eigenvalues and eigenvectors provide the solution (we may have deficient eigenvalues, but the ideas are very similar in this case as well). Differential equations that model competition between species for resources (rabbits vs. squirrels) or predator-prey systems (foxes vs. rabbits) may have the same form as the difference equation we solved above. The ideas behind eigenvalues and eigenvectors can be extended even further. Vibrations of a guitar string or a drumhead can be modeled using equations involving partial derivatives (partial differential equations). The solution to such equations can be found using eigenvalues and eigenfunctions (since we are dealing with functions instead of vectors in $\mathbb{R}^n$, we use eigenfunctions instead of eigenvectors–the ideas are still very similar).

Problem 2 was taken from

Bretscher, Otto. *Linear Algebra with Applications, 3rd edition.* Upper Saddle River: Pearson Prentice Hall, 2005.

In particular, I used material from sections 7.1, 7.5, and 7.6.