Foundations of Analysis II
Week 14

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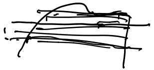
Lebesgue Measure

- Extend length of intervals to $m : 2^{\mathbb{R}} \to [0, \infty]$:

$f(\{x : f(x) = a\})$



First try: $m : 2^{\mathbb{R}} \to [0, \infty]$

1) $m([a,b]) = b - a$

2) $\{A_i\}$ $A_i \cap A_j = \emptyset$ for $i \neq j$

$\Rightarrow m\left(\bigcup_i A_i\right) = \sum_i m(A_i)$

$\leq$

countable additivity

$A \subset B$

$m(A) \leq m(B)$

3) $m(A + x) = m(A)$ $\forall x$

translation invariance

Not possible

$\exists\ \underline{P} \subset [0,1)$

$[0,1) \to S^1$

$+ : [0,1)\times[0,1) \to [0,1)$

$\Leftrightarrow$ add of xs

$(x+y) \pmod 1$

$[0,1)$ $x \sim y$

$\Leftrightarrow x - y \in \mathbb{Q}$

Axiom: choose one element from equiv class

$\underline{P}$

$\{t_i\} = \text{[illegible]}\ \mathbb{Q}$

$\exists$ Countable $\{t_i\}_{i=1}^{\infty}$

s.t. $(P + t_i) \cap (P + t_j) = \phi$

$\bigcup_{i=1}^{\infty} P + t_i = [0,1)$

$\underline{m(P)} = \begin{cases} 0 \\ a > 0 \end{cases}$

$m([0,1)) = \sum 0 = 0$

$m([0,1)) = \sum_{i=1}^{\infty} a = \infty$

mean = length.

P: $\underline{\text{non-measurable}}$ set

σ-algebras

Counting meas

$m E = \infty$ if E ∞

$\# E$ if E finite

measure defined on $2^{\mathbb{R}}$, $m[a,b] = \begin{cases} 1 & \text{if } a=b \\ \infty & \text{if } a<b \end{cases}$

σ-algebra closed under complements, countable unions

$\emptyset$.

$\Downarrow$ countable intersections

Want: a σ-alg $\mathcal{M}$ = measurable sets

Measure

$$m : \mathcal{M} \longrightarrow [0, \infty]$$

Countable additivity

$$m\left(\bigcup A_i\right) = \sum m(A_i)$$

$$\{A_i\} \subset \mathcal{M} \quad \text{s.t.} \quad A_i \cap A_j = \phi \quad \forall\, i \neq j$$

$$m(A \cup B) = m(A) + m(B)$$

$$(A \cap B^c)$$

$$A \subset B$$

$$\Rightarrow m(A) \leq m(B)$$

$$B = A \cup (A^c \cap B)$$

Monotone

$$m(B) = m(A) + \underbrace{m(A^c \cap B)}_{\geq 0}$$

Outer Measure m^*

$A \subset \mathbb{R}$

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \ell(I_i) : \{I_i\} \text{ open intervals s.t. } A \subset \bigcup I_i \right\}$$

defined $\forall A \subset \mathbb{R}$

$$m^*([a,b]) = b - a$$

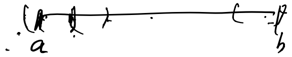
$[a,b] \subset (a-\varepsilon, b+\varepsilon) \quad \forall \varepsilon > 0$

$\Rightarrow m_*[a,b] \le (b-a) + 2\varepsilon \quad \forall \varepsilon > 0$

$$\boxed{m_*[a,b] \le b-a}$$

any $\{I_i\}$ $\quad \underbrace{\sum \ell(I_i)}_{} \ge b-a$

Compactness $\quad [a,b] \subset \underline{I_{i_1} \cup \cdots \cup I_{i_k}}$



$\Rightarrow \sum \ell(I_i') \ge b-a$

$a_1 < a < b_1$

$a_2 \cdots b_2$ ——

Countable sub-additivity

- $m^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} m^*(A_i)$ given $\varepsilon > 0$

i $\{I_{i,j}\}$ cover A_i $\sum \ell(I_{i,j}) < m_*(A_i) + \varepsilon/2^i$

$\bigcup_{i,j} I_{ij}$ cover $\cup A_i$.

$m(\bigcup A_i) \leq \sum_{i,j} I_{ij} < \sum m_*(A_i) + \varepsilon$

$\forall \varepsilon > 0$ $m(\cup A_i) \leq \sum m_*(A_i) + \varepsilon$

Measurable sets

Carathéodory

Definition

Lebesgue

$E \subset \mathbb{R}$ is *measurable* $\iff \forall A \subset \mathbb{R}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

E

σ-algebra

$\mathcal{M} \subset 2^{\mathbb{R}}$

s.t. $m^* \big|_{\mathcal{M}}$ is countably additive

m^* defined $\forall A \subset \mathbb{R}$

translation inv

$m^*[a, b] = b - a$

countable sub-additive

A meas

$m_L(A) = m^*(A)$

Lebesgue m^* outer

m_* inner

Theorem

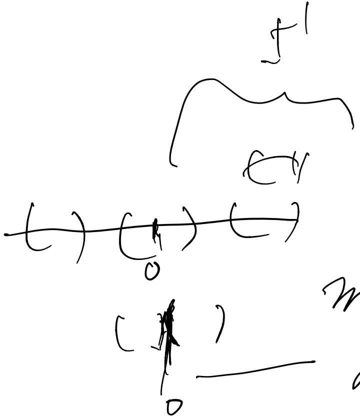
$\mathcal{M}$ is a σ-algebra.

Royden Lemma 6 – 9

Thm 10

$m^*(A) \le m_*(A \cap E) + m_*(A \cap E^c)$

measurable



prove $(0,\infty)$ is measurable.

$A \subset \mathbb{R}$

$m^*(A) \geq m^*(A \cap (0,\infty)) + m^*(A \cap (-\infty, 0])$

$m^*(A)$

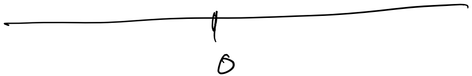
any $\{I_i\}$ covering A, =

$\subset (0,\infty)$ or

$\{I_i\} \to \{I'_j\}$

$\{I''_k\}$

$\sum \ell(\cdot)$



$0 \in I_i$ or $0 \notin I_i$

$I_i \to I'$ a covering of A, $0 \in I$

$I_i \to I''$

$I \cap (0,\infty)$

$\cup I_i = \cup I^1 \cup \cup I^2$

$\Sigma \ell(I_i)$

$= \Sigma \ell(I_i^1) + \Sigma \ell(I_i^2)$

Σ

Lemma

(Royden)

- $[0, \infty) \in \mathcal{M}$

$\mathcal{B}$ = Borel sets

def smallest σ-alg containing open sets.

$\mathcal{M} \ni (a, \infty)$

$\ni (-\infty, b]$

$\Rightarrow \mathcal{M} \supset$ all open sets

$\supset$ all closed sets

- $\mathcal{B} \subset \mathcal{M}$

Regularity

E measurable $\iff \forall \varepsilon > 0\ \exists$ open set O, $E \subset O$, $m_*(O) < m_*(E) + \varepsilon$

$\exists$ closed set F, $F \subset E$ s.t. $m_*(F) \geq m(E) - \varepsilon$

$$\begin{cases} m^* = \inf\{m(O) : O \text{ open} \supset E\} \\ m_* = \sup\{m(F) : F \text{ closed} \subset E\} \end{cases}$$

sets of type G_δ = countable $\cap$ of open sets

F_σ = countable $\cup$ of closed sets

G = Gebiet $\quad \delta$ = durchschnitt

F = Fermé $\quad \sigma$ = summe

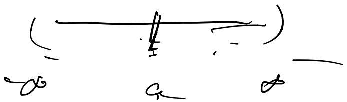
Measurable Functions

$E \subset \mathbb{R}$

$f : E \to \mathbb{R}$ is meas $\iff$ E measurable

$\forall a \in \mathbb{R}$

$f^{-1}(a, \infty]$ is meas.



$\iff f^{-1}(-\infty, a], \ldots$

$[a, \infty)$

$[a, \infty]$

$[-\infty, a]$

$(-\infty, a)$

E meas

$\iff \exists\, G \in G_\delta$

$F \in F_\sigma$

s.t. $F \subset E \subset G$

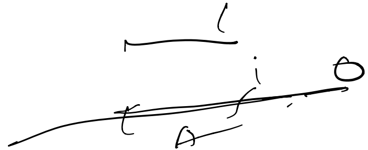
$m(G - E) = 0$

$m(E - F) = 0$

Cont $\rightarrow$ Maxwell

Characteristic Functions, Simple Functions

$A \in \mathcal{M}$ $\qquad \chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$



Simple: $\sum_{\text{finite}} a_i \chi_{A_i}$

Recall Lebesgue Measure

give up

- There is no measure $m : 2^{\mathbb{R}} \to [0, \infty]$ which is
  - Countably additive
  - $m([a, b]) = b - a$ for all intervals $[a, b] \subset \mathbb{R}$. What
  - Translation invariant.
- Defined $\mathcal{M} \subset 2^{\mathbb{R}}$, the *Lebesgue measurable sets*
  - Outer measure $m^* : 2^{\mathbb{R}} \to [0, \infty]$
  - $m*$ is translation invariant, $m^*([a, b]) = b - a$
  - But m^* is only countably sub-additive.
- Define
  $\mathcal{M} = \{E \subset \mathbb{R} : m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)\}$

$m^*(A) = \inf \{ \sum l(I_i) : \{I_i\}$ open $A \subset \cup I_i \}$

$m^*(A) = 0$ set of measure zero

$\forall \varepsilon > 0 \ \exists \{I_i\} \ \sum l(I_i) < \varepsilon$

$\Leftrightarrow m^*(A) = 0$

Thm $m^*[a,b] = b-a$

countably $m^*(\cup A_i) \leq \sum m^*(A_i)$
sub-additive.

~~Want.~~

Lebesgue meas set $\mathcal{M}$

$E \in \mathcal{M}$

$\Leftrightarrow \forall A \subset \mathbb{R}$

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

$\leq$ clear..

$\geq$

$\mathcal{M}$ is a σ-algebra (work, see Royden)

$A \in \mathcal{M}$, let $m(A) = m^*(A)$

$m^*|_{\mathcal{M}}$ = measure m

$m(\cup A_i) = \sum m(A_i)$ if $A_i \cap A_j = \phi$ $\forall i \neq j$

$m : \mathcal{M} \to [0, \infty)$

translation inv, $m[a,b] = b-a$

Lebesgue measurable sets

- $\mathcal{M}$ is a σ-algebra. (Royden Thm 10)
- $m^*(E) = 0 \Rightarrow E \in \mathcal{M}$
- If $m(F) = 0$, then $E \in \mathcal{M} \iff E \cup F \in \mathcal{M}$
- $(0, \infty) \in \mathcal{M}$ (Royden Lemma 1).
- $\mathcal{B} \subset \mathcal{M}$ where $\mathcal{B}$ = Borel sets, the smallest σ-algebra containing all open sets.
- Regularity
- Measure continuity.

(a, ∞)

$[a, \infty)$

$(-\infty, b)$

$(-\infty, b]$

$\{A_i\} \quad A_i \cap A_j = \phi \quad \forall i \neq j$

$\exists m(\bigcup A_i) = \sum m(A_i)$

A_i find A_i' $\quad \bigcup A_i' = \bigcup A_i$

$A_i' \cap A_j' = \emptyset \quad i \neq j$

$A_1' = A_1$

$A_2' = A_2 - A_1$ $\qquad m(A_i) \geq m(A_i')$

$A_3' = \underline{A_3} - (\underline{A_1 \cup A_2})$ $\qquad A_i = A_i' \cup (\ldots)$

$\vdots$ $\qquad m(A_i) = m(A_i') + m(\ )$

$A_i' = A_i - (A_1 \cup \cdots \cup A_{i-1})$

$\vdots$

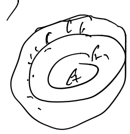
(A_i') dis, $m(\bigcup A_i) = m(\bigcup A_i') = \sum m(A_i')$

$A_i \in \mathcal{M}$

$A_1 \subset A_2 \subset A_3 \subset \cdots$

$A = \bigcup A_i$ $\qquad \sum m(A_i')$

$m(A) = \lim_{i \to \infty} m(A_i)$



$m A_i = \sum_{j \leq i} m(A_j')$

Measurable Functions

f measurable

$\Leftrightarrow$ 1) dom is measurable

2) $\forall a \in \mathbb{R}$, $f^{-1}(a, \infty) \in \mathcal{M}$

$\Leftrightarrow$ $f^{-1}(-\infty, b) \in \mathcal{M}$

⋮

$\forall$ open f^{-1}(open) meas.

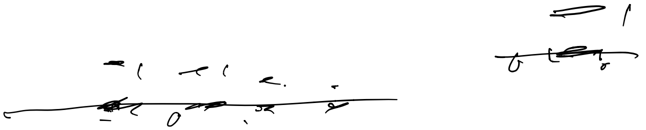
Characteristic Functions, Simple Functions

$A \subset \mathbb{R}$

char function of A:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

χ_A meas $\iff$ A measurable



Simple func

1) a finite sum of charac fns

$\sum_{i=1}^{n} a_i \chi_{A_i}$

2) a func with only finitely many values.

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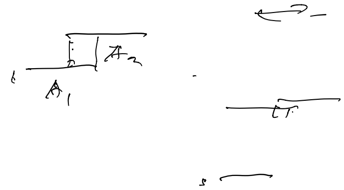
f values $a_1, \dots, a_n$

$A_i = \{x : f(x) = a_i\}$

$f = \sum a_i \chi_{A_i}$ → canonical presen of f as a linear comb of charac func.

$\sum a_i \chi_{A_i} = \sum b_j \chi_{B_j}$



$A_i \cap A_j = \phi$ if $i \neq j$

define $\int f$

$\int \chi_A = m(A)$

1) χ_A measurable $\Longleftrightarrow$ A measble.

simple func measble $\Leftrightarrow$ A_i in canonical presen are meas

$$\boxed{f = \sum a_i \chi_{A_i} \Rightarrow \int f = \sum a_i m(A_i)}$$

Lemma 1

Boyden Chp 4

is inside of boundary
of ~~the~~ as a linear comb of χ_{A_i}

Ro

The Lebesgue Integral

- Integral of simple function

- Recall Riemann integral

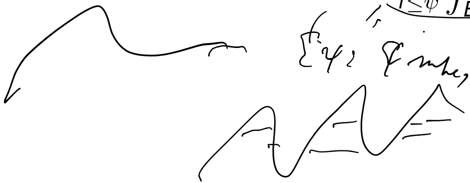
Prove this

(Royden Chap 4 Prop 3)

- $f : E \to \mathbb{R}$ bounded, E measurable, $m(E) < \infty$
- f is measurable iff

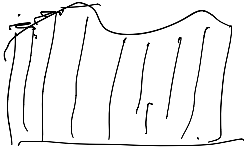
$$\inf_{f \le \psi} \int_E \psi(x)dx = \sup_{f \ge \phi} \int_E \phi(x)dx$$

$\{\psi : \psi \text{ simple}, \psi \ge f\}$ $\quad$ $\{\varphi : \varphi \text{ simple}, \varphi \le f\}$



Either one is called the Lebesgue integral of f.

Compare with Riemann.



partition P

$U(f, P)$ upper sum

$L(f, P)$ lower sum

$$L \le U$$

$$R\overline{\int_a^b} f\,dx = \inf \{U\}$$

$$R\underline{\int_a^b} = \sup \{L\}$$

$U = \int$ simple fn.

$$L \le \sup_{f \ge \varphi} \int f \le \inf_{\{f \le \psi \text{ simple}\}} \int \psi(x)\,dx \le \inf U(f,P)$$

f is Riemann int, then $\sup_P L = \inf_P U$.

$$\Rightarrow \sup_{\{f \ge \varphi\}} \int \varphi = \int_{\{f \le \psi\}} \psi .$$

Riemann int $\Rightarrow$ Lebesgue meas and R-int $= \mathcal{L}$-integral

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$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$$

not R-int
but $\mathcal{L}$-int $\mathcal{L}\int = 0$

$f = 0$ a.e, almost everywhere

- Recall Royden Chp 4, Prop 3:
- $f : E \to \mathbb{R}$ bounded, E measurable, $m(E) < \infty$.
- Then f is measurable iff

$$\inf_{\psi \text{ simple},\, f \le \psi} \int_E \psi(x)dx = \sup_{\phi \text{ simple},\, f \ge \phi} \int_E \phi(x)dx$$

where ϕ, ψ are simple functions.

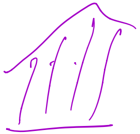
Def of Lebesgue int.

$$\int_E f \, dm = \sup_{\substack{s \text{ simple} \\ s \le f}} \int s.$$

Picture

$M \geq \|f\|$

- The sets $E_{n,k} = \{x : \frac{kM}{n} \geq f(x) > \frac{(k-1)M}{n}\}$



- The simple functions
  - $\psi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} k\, \chi_{E_{n,k}}(x)$ and   above
  - $\phi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\, \chi_{E_{n,k}}(x)$ below

$\inf_{\psi \geq f} \int - \sup \int \leq \int \psi_n - \int \varphi_n \leq \frac{M}{n} \sum_{-n}^{n} m(E_{n,k})$

$\inf \int_{\psi \geq f} \leq \int \psi_n$

$\sup \int_{\varphi \leq f} \geq \int \varphi_n$

$\inf - \sup \leq \int (\psi_n - \varphi_n) = \frac{M}{n} \sum m(E_{n,k}) = \frac{M}{n} m(E) \to 0 \text{ as } n \to \infty$

$\leq U \ldots \leq U \int_a^b$

$R\int \leq \mathcal{L}\int \leq \mathcal{L}\int \leq R\int$

$\Rightarrow$

Riem int $\Rightarrow$ $R\int = \mathcal{L}\int$

$f \geq 0$ on any E

$\sup_{E'} \int_{E'} f$

The Lebesgue Integral

$\int_E f$ bdd, $m(E) < \infty$

$\longrightarrow \int_E f \quad f \geq 0,\ m(E)$ arbitrary

$\longrightarrow \int_E f \quad$ by $\quad f = f^+ - f^-$

$\int f+g = \int f + \int g \qquad \int_E f = \int_E f^+ - \int_E f^-$

$f(x) = \begin{cases} 1 & \text{on } \mathbb{Q} \\ 0 & \text{on } \mathbb{R} - \mathbb{Q} \end{cases}$ on $[0,1]$ $\quad$ ~~Not~~ not R-int, but $\mathcal{L}$ - cai

$\mathcal{L}\int_0^1 f = 0$

$\mathcal{R}\int_0^1$ not defined

$\int_0^1 = 1 \qquad \int_0^1 = 0$ Example

$1_{\{x : f(x) = 1\}} \quad m(A) = 0$

a.e.

Rudin's variation

- $E_{n,i} = \{x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\}$
- $F_n = \{x : f(x) \geq n\}$
- $s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\,\chi_{F_n}$

Monotone Convergence Theorem

$0 \leq f_1 \leq f_2 \leq \cdots$ $\quad f_n \to f$ ptwise

$\Rightarrow \int f_n \to \int f$.

$f = 0$ on $\mathbb{Q}$, 1 $\mathbb{R}-\mathbb{Q}$

f_n.

$f_1 \leq f_2 \leq f_3 \cdots \to f$

$\int f_n \to \int f$ $\quad f_n \leq f$

$\int f_n \leq \int f$

$=$

$\int f_n \to \alpha = \sup\{\int f_n\} \geq \int s$

$\int_E \quad \forall s$

$\sup \int f_n \geq \int f$

$C \to 1$

$\int f_n \leq \int f$

$\Rightarrow \alpha \leq \int f$

Need $\int f \leq \alpha$

$s \leq f$

$\sup \int s$

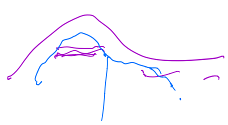
$0 < c < 1$

Fix $s \leq f$, c $\quad 0 < c < 1$

$cs < f$

f_n

~~$x : f_n(x) \geq$~~



$f_n \to f$

$s < f$

$\forall x \ \exists n$ s.t. $f_n(x) > c\, s(x)$

$E_n = \{x : f_n(x) \geq c\, s(x)\}$

$\bigcup E_n = E$

$\int_{E_n} cs \xrightarrow{?} \int_E cs$

$\int_{E_n}$

Fact

$f \geq 0$, meas,

$X \subset \mathbb{R}$

$E \to \int_E f = \int_X \chi_E f$

$E \to \int_E f$ is countably additive

$E_i \cap E_j = \emptyset \quad i \neq j$

$\int_{\bigcup E_i} f = \sum \int_{E_i} f$

$\Rightarrow E = \bigcup E_i, \quad \int_E f = \lim \int_{E_i} f$

$\int_{E_n} f_n(x) \geq c \int_{E_n} s \to c \int_E s$

Fatou's Lemma

$\{f_n\}$ $f_n \geq 0$, meas fns on E

Rudin
proof
from
Monotone Convergence

$$\int \underline{\lim} f_n \quad \underline{\lim} \int f_n$$

$0 \qquad 1/2$

$\underline{\lim} f_n = 0 \qquad \int f_n = 1/2$

$$\int \underline{\lim} f_n \leq \underline{\lim} \int f_n$$

$$f_n \longrightarrow g_n$$

$$g_n(x) = \inf\{f_n(x), f_{n+1}(x), \ldots\}$$

$$g_1 \le g_2 \le \cdots$$

$$g_n \le f_n$$

$$\lim g_n = \underline{\lim}\, f_n$$

$$\int g_n \to \int \underline{\lim}\, f_n \qquad \int g_n \le \int f_n$$

$$\underline{\lim} \int g_n \le \underline{\lim} \int f_n$$

$$\int f_n \le \int \underline{\lim}\, f_n$$

$$\boxed{\int \underline{\lim}\, f_n \le \underline{\lim} \int f_n}$$

Dominated Convergence Theorem

$f \leq g$ a.e.

$$\int f \leq \int g$$

$$\int |f| = \int |f|$$

$\mathcal{L}(E)$ $\mathcal{I}$ mu

$$\int |f| < \infty$$

f int $\Rightarrow$ $|f|$ int.

f^+, f^-

$\int f = \int f^+ - \int f^-$

$\int f^+ + f^- = \int |f|$

$f_n \to f$ pw

$\forall \varepsilon$ m $|f_m(x) - f(x)| < \varepsilon$

$X = X_1^{(\varepsilon)} \cup X_2^{(\varepsilon)}$

$m(X_2) < \delta$

$|f_n(x) - f(x)| < \varepsilon$