Foundations of Analysis II
Week 2

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Spring 2019

Compactness & Sequential Compactness

Compact

Every open cover has finite sub-cover

Given any collection $\{U_\alpha\}_{\alpha \in A}$ of open sets in X

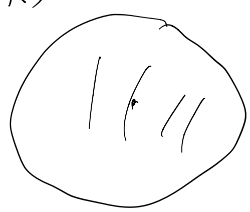
s.t. $K \subset \bigcup_{\alpha \in A} U_\alpha$

$\exists \alpha_1, \dots, \alpha_n \in A$

s.t. $K \subset \bigcup_{i=1}^{n} U_{\alpha_i}$

Thm $K \subset X$ compact $\Longrightarrow$ K is bounded

$\exists x, R$ s.t. $K \subset B(x, R)$



Pf take x_0, say $x_0 \in K$

$A = \mathbb{R}^{>0} = \{r \in \mathbb{R} : r > 0\}$

$U_r = B(x_0, r)$

$$\bigcup_{r \in \mathbb{R}^{>0}} U_r = X$$

K cpt $\Rightarrow \exists r_1, \dots, r_n$

s.t. $K \subset U_{r_1} \cup \dots \cup U_{r_n}$

$= U_{r_{\max\{1,\dots,n\}}}$

$= B(x_0, r_{\max\{1,\dots,n\}})$

$r_{\min}$

$K \subset X$ compact $\Rightarrow$ K is closed

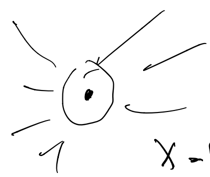
$\Updownarrow$

$X \setminus K$ open

$\forall x_0 \notin K$

$\exists r > 0$ s.t. $B(x_0, r) \cap K = \phi$

$V(x_0, r) = \{x \in X : d(x, x_0) > r\}$



$\bigcup_{0 < r < \infty} V(x_0, r)$

$X - \{x_0\}$ ''

$x \in X - \{x_0\}$

$\Leftrightarrow$ $x \neq x_0$

$\Leftrightarrow$ $d(x, x_0) > 0$

$\Leftrightarrow$ $\exists r > 0$ s.t. $d(x, x_0) \geq r$

$\bigcup \underline{V(r, x_0)} = X - \{x_0\} \supset K$

K open $\Rightarrow$ finite subcov

$r_1, \dots, r_n$ s.t.

$K \subset \underline{V(r_1, x_0)} \cup \dots \cup V(r_n, x_0)$

r_{max} $\quad B(x_0, r_{max}) \cap K = \phi$

Compact $\Rightarrow$ closed, bounded

in $\mathbb{R}^n$ converse true.

---

in $C([0,1])$ closed, bdd $\not\Rightarrow$ compact.

Thm $f : K \to \mathbb{R}$ cont,

K compact

$\Rightarrow f$ uniformly cont.

Sequential compactness

$K \subset X$ is seq. compact

$\Leftrightarrow$ every sequence $\{x_n\}$ in K has a subsequence that converges to some $x \in K$.

Fact They are equiv (in any metric space)

Compactness $\Rightarrow$ Seq compactness

easy

$\Leftarrow$

harder

not cpt $\Leftarrow$ not Seq Compact

$\exists$ seq $\{x_n\}$ no conv subseq

Set values $\{x_n\}$

∞, no limit pts $\Rightarrow$ closed

Seq Compact ⇒ Compact

See Rudin, Chapter 2, Exercise 26.

Completeness

Compactness.

Equicontinuity

$\mathcal{F} = \{f\}$ on X is equicontinuous

$\Leftrightarrow$ $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon)$

s.t. $d(x,y) < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon$

Thm in $C(X)$ X cpt metric space

(C ex: $C[0,1]$)

a set is compact $\Leftrightarrow$ closed

bdd

equicontinuous

Examples

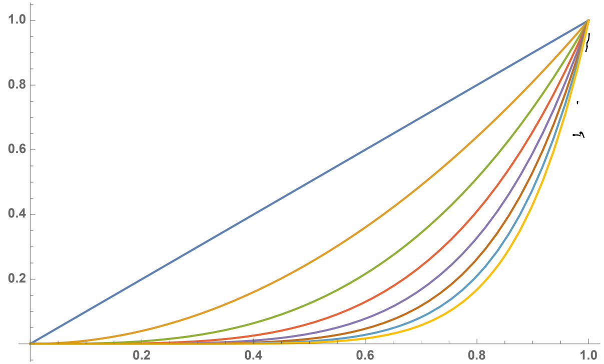
HW

here

$\{f_n\}$ $\{f_n'\}$ conv, unif bdd

x^n $[0,1]$

Out[•]=



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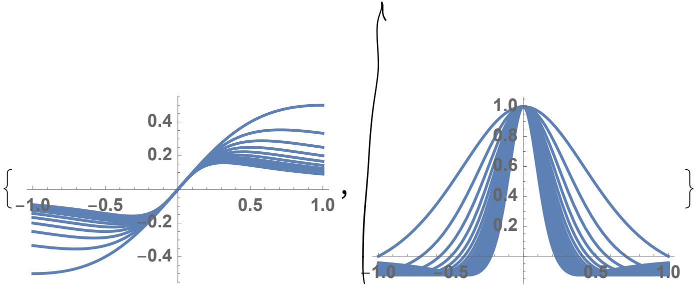


$\{f\}$

$\{f'\}$ cont., $\|f''\|_\infty$ bdd.

$\exists\, C$ s.t. $|f'(x)| < C$

$\forall x, \forall f$

$|f(x) - f(y)| = \left| \int_x^y f'(t)\,dt \right| \leq |x$

$$\left|\int_y^x f'(s)\,dt\right|$$

$$\leq \int_x^y |f'(t)|\,dt$$

$$\leq \underbrace{}_{\leq C}$$

$$\underline{C\,|x-y|}$$

$\forall \varepsilon > 0 \quad \delta = \varepsilon / C$

$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$
$\forall x, y, f$

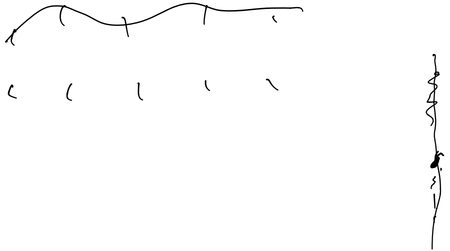
To prove $\quad X$ $\underline{\text{compact}}$ metr.

$C(X)$

closed, bdd, equicont $\Rightarrow$ compact

Theorem

- $E = \{x_1, x_2, \dots\}$ *a countable set.* $\forall x \in E \ \exists \ C(x) \ |f_n(x)| < C \ \forall n$
- $f_n : E \to \mathbb{R}$ *pointwise bounded sequence of functions.*
- *Then* $\{f_n\}$ *has a subsequence* $\{f_{n_k}\}$ *that converges at every* $x \in E$.



Pf $\{f_n\}$

$\{f_n(x_1)\}$ bdd seq in $\mathbb{R}$ $\Rightarrow$ has conv subseq $f_{n_k}(x_1)$

$\{f_{n_k}\}$ is a sub-seq of $\{f_n\}$ s.t. $f_{n_k}(x_1)$ conv.

$f_{n_k}(x_2)$ bdd seq of ——

$\exists$ sub seq $\{f_{n_{k_\ell}}\}$ of $\{f_n\}$ of $\{$—

so $f_{n_{k_\ell}}(x_1)$ & $f_{n_{k_\ell}}(x_2)$ conv.

$\vdots$

Recall: f_{n_k}

$f_{1,k}$ sub. of f_n conv. at x_1

$f_{n_{k_\ell}}$ $f_{2,\ell}$ - - - - - - x_2 & x_1

Conv x_1 $\longrightarrow$ $f_{1,1}$ $f_{1,2}$ $f_{1,3}$ $f_{1,4}$ $f_{1,5}$.

subseq

Conv at x_2 $\longrightarrow$ $f_{2,1}$ $f_{2,2}$ $f_{2,3}$. $f_{2,4}$

Subseq
Conv at x_3 $f_{3,1}$ $f_{3,2}$ $f_{3,3}$. — —

$g_{n,n}$

diagonally $f_{m,m}$

Subseq of orig.

Conv. at $x_1, x_2, \ldots$

$f_{m,1}$ $f_{1,\ldots}$ $f_{n_k,m}(x_j)$ in $f_{1,k}$ $k \geq m$

Theorem

- X compact metric space
- $f_n \in \mathcal{C}(X)$, $n = 1, 2, \ldots$
  - pointwise bounded
  - equicontinuous
- then
  - $\{f_n\}$ is uniformly bounded
  - $\{f_n\}$ contains a uniformly convergent subsequence

[Seq compress of closed, bdd, equicont sets)

$\delta > 0$ $\exists\, y_1, \ldots, y_{n_\delta}$ s.t. $\forall x \in X$, $\exists\, y_i$ s.t. $d(x, y_i) < \delta$

X cpt metric $\Rightarrow$ countable dense set $E = \{x_1, x_2, \ldots\}$

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Due Jan ~~26~~

HW: Jan 23

Recall:

Theorem

- $E = \{x_1, x_2, \dots\}$ *a countable set.*
- $f_n : E \to \mathbb{R}$ *pointwise bounded sequence of functions.*
- *Then* $\{f_n\}$ *has a subsequence* $\{f_{n_k}\}$ *that converges at every* $x \in E$.

---

conv $x_1 \xleftarrow{\text{subs}} f_{1,1}, f_{1,2}, f_{1,3}, \dots$ $\quad S_1$

$x_2 \quad$ subs $f_{2,1}, f_{2,2}, \dots$ $\quad S_2$

$x_3 \quad \vdots \quad S_3$

$\vdots$

$f_{1,1}, f_{2,2}, f_{3,3}, f_{4,4}, \ldots$ subset of f_n

for each k, $\deg(f_j, j)$ is a subseq of S_k if $j \geq k$

$\Rightarrow \deg(j,j) \to \infty$ at $x_1, \ldots, x_k$

Theorem

Thm 7.25

- X compact metric space
- $f_n \in \mathcal{C}(X)$, $n = 1, 2, \ldots$
  - pointwise bounded
  - equicontinuous
- then
  1. $\{f_n\}$ is uniformly bounded
  2. $\{f_n\}$ contains a uniformly convergent subsequence

bdd, equicont set in $C(X)$ has compact closure

$\Rightarrow$ closed, bdd, equicont $\Rightarrow$ compact

$\overleftarrow{\Rightarrow}$ [illegible]

Pf of ①

X cpt

f_n $\cancel{(x)}$ bdd $\forall x$

$|f_n(x)| \le C(x)$

$\forall n$

$\exists C$

$\Longrightarrow |f_n(x)| \le C \quad \forall n, \forall x$

max C_x ??

find finite $x_1, \dots, x_m$

Hint: ① $\boxed{X \text{ cpt}, \ \delta > 0}$ $\exists x_1, \dots, x_m \in X$

s.t. $\forall x \in X \ \exists x_i$

s.t. $d(x, x_i) < \delta$

what open cover of X?

② $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $d(x,y) < \delta$

$\Rightarrow |f_n(x) - f_n(y)| < \varepsilon$

$\forall x, y, f_n$

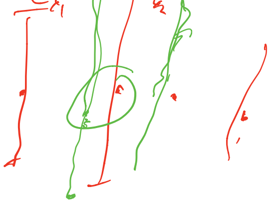
$\varepsilon = 1 \quad \exists \delta$

$x_1, \ldots, x_n \quad C' = \max C_{x_1}, \ldots, C_{x_n}$

$|f_n(x)| = |f_n(x) - f_n(x_c) + f_n(x_c)|$

~~$\exists \delta$ s.t. $f_n(x) - f_n(\ldots)$~~ $\leq |f_n(x) - f_n(x_c)| + |f_n(x_c)|$

$\delta \quad d(x,y) < \delta \Rightarrow |f_n(x) - f_n(y)| < 1 \quad \forall f_n, x, y.$



$x \in X$

$x \quad |f_n(x) - f_n(\text{some } x_i)| < 1$

because $d(\text{some } x_i, x) < \delta$

$|f_n(x)| \leq C' + 1$

(2) X compact, metric $\Rightarrow \exists$ countable dense set $E \subset X$

$\{x_1, x_2, \ldots\}$

$1/n$ $\exists\, x_{n,1}, \ldots, x_{n,k_n}$

st $\forall x \in X\ \exists\, x_{n,i}$

such that $d(x, x_{n,i}) < 1/n$

n $\underbrace{x_{n,1} \ldots x_{n,k}}_{}$ E_n

Take $E = \bigcup E_n$

$f_n \to$ subseq that conv on E

$\{g_i\}$

$\varepsilon > 0\ \exists N$ st $i, j > N \Rightarrow$

$|g_i(x) - g_j(x)| < \varepsilon \quad \forall x$

Stone-Weierstrass Theorem

Weirstrass Theorem:

any int $[a,b]$

Theorem

If $f \in \mathcal{C}([0,1])$, then there exists a sequence $\{P_n\}$ of polynomials such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly in $[0,1]$. *(equivalently,* $||P_n - f|| \to 0$*)*

Note: if f is "real analytic function"

$=$ its ∞ Taylor Series $\to f$

not

note: f is not assumed diff

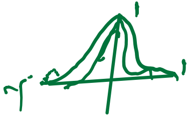
Pf in Rudin

(3) may assume $f(0)=f(1)=0$

(replace f by $f-(ax+b)$ "linear fcn" $=f(0)$ at $x=0$, $f(1)$ at $x=1$)

Look at functions

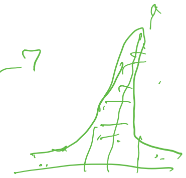
$(1-x^2)^n$ on $[-1,1]$

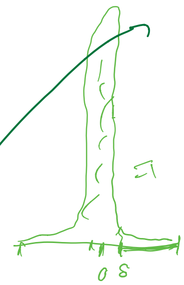


Let $Q_n(x) = c_n(1-x^2)^n$

where $c_n = \frac{1}{\int_{-1}^{1}(1-x^2)^n\,dx}$

So $\boxed{\int_{-1}^{1} Q_n(x)\,dx = 1}$





① $\int_{-1}^{1} Q_n(x)\,dx = 1$

② $\forall \delta > 0$, $Q_n(x) \to 0$ unif on $\delta \le x \le 1$

Dirac's δ-func: $\boxed{0 \text{ if } x \neq 0,\ \infty \text{ if } x = 0,\ \text{whose int} = 1}$

extend f to $\mathbb{R}$ by $f(x) = 0$ if $x \notin [0,1]$

Def of $P_n(x)$:

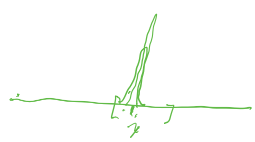
$$\boxed{P_n(x) = \int_{-\infty}^{\infty} f(x+t)\,Q_n(t)\,dt}$$

Claim

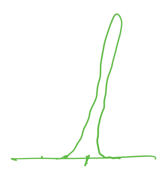
(1) $P_n(x)$ poly in x

(2) $\boxed{P_n \to f \text{ unif on } [0,1]}$

Picture



$\int_{-\infty}^{\infty} f(x+t)\,Q_n(t)\,dt \to f(x)$



$f(x+t) - f(x)$

$\int_{-\infty}^{\infty} f(x+t)\,Q_n(t)\,dt \to f(x)$

$$\int \left(f(x+t) - f(x) \right) Q_n(t)\, dt$$

f cont: $\forall \varepsilon > 0\ \exists\ \delta \quad |f(x+t) - f(x)| < \varepsilon$, $|t| < \delta$

$$\left| \int_{-\infty}^{\infty} f(x+t) - f(x) Q_n(t)\, dt \right|$$

$$\leq \int_{-\infty}^{\infty} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

$< \varepsilon$ for $|t| < \delta$

$$\leq \varepsilon \int Q_n(t)\, dt = 1$$

$1 \leq 0$ $x+t \leq 1$ $-x < t < 1-x$

Q small, Q big and $\int Q \sim 1$

Q small

$-x$ $-\delta$ $1-x$

$\boxed{\text{Why is } P_n(x) \text{ poly?}}$

$$\int_{-\infty}^{\infty} f(x+t)\, Q_n(t)\, dt$$

$\boxed{s = x+t}$ change of variables

$t = s - x$

$$= \int_{-\infty}^{\infty} f(s)\, Q(s-x)\, ds = \text{poly in } x$$

$Q_n(s-x)$

$$= \frac{\left(1 - (s-x)^2\right)^n}{c_n}$$ expand

$=$ poly in s, x

$\int$ (poly in s, x) $ds =$ poly in x

$\sum - s^{(\cdot)} x^{(\cdot)}$

$x^2 \le 1$

$$c_n = \int_{-1}^{1} (1-x^2)^n \, dx$$

$$Q_n(x) = \frac{(1-x^2)^n}{c_n}$$

$$\boxed{\int_{-1}^{1} Q_n(x)\, dx = 1}$$

$(1-x^2)^n \geq 1 - nx^2$ on $[-1,1]$

$$1/c_n = \int_{-1}^{1} (1-x^2)^n\,dx$$

$$\geq 2\int_0^1$$

$$\geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n\,dx$$

$$\geq 2\int_0^{1/\sqrt{n}} (1-nx^2)\,dx$$

$$x - x^3/3\Big|_0^{1/\sqrt{n}} = 4/3\,\frac{1}{\sqrt{n}} > 1/\sqrt{n}$$

$$\boxed{c_n < \sqrt{n}}$$

$$\boxed{Q_n(x) = c_n(1-x^2)^n \quad \text{wn}}$$ $\int_{-1}^{1} Q_n(x)\,dx = 1$

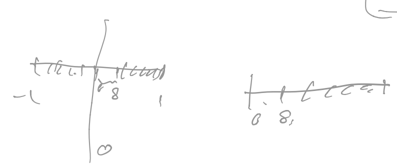
Estimate $0 \leq Q_n(x) \leq \sqrt{n}(1-\delta^2)^n \longrightarrow 0$ as $n \to \infty$

on $[\delta, 1]$

$\sqrt{n}(1-\delta^2)^n$

$\sqrt{n} \to \infty$

$(1-\delta^2)^n \to 0$



$$P_n(x) = \int_{-x}^{1-x} f(x+t)\,Q_n(t)\,dt$$

Given $\varepsilon > 0$, choose $\delta > 0$ s.t. $|y-x| < \delta$
$\Rightarrow |f(y) - f(x)| < \varepsilon$

$M = \sup_{x\in[0,1]} |f(x)|$ $\boxed{\text{unif cont of } f}$

$$|P_n(x) - f(x)| = \left|\int_{-1}^{1} (f(x+t) - f(x))\,Q_n(t)\,dt\right|$$

$$\leq \int_{-1}^{1} |f(x+t) - f(x)|\,Q_n(t)\,dt.$$

$2M\int_{-1}^{\delta}$ $\int_{-\delta}^{\delta}$ $\int_{\delta}^{1}$

$2M\int_{-1}^{\delta} Q_n(t)\,dt$ $\quad \varepsilon\int_{-\delta}^{\delta} Q_n\,dt$ $\quad 2M\int_{\delta}^{1} Q_n\,dt$

$\sum 4m\sqrt{n}(1-\delta^2)^n + \varepsilon \to 2\varepsilon$

(p. 160)

http://www.math.tifr.res.in/~publ/ln/tifr16.pdf

Good reference for the Weirstrass thm.

Chap 1

Now prove $P_n(x) \to f(x)$ unif on $[0,1]$

$Q_n(x) = c_n(1-x^2)^n$

c_n chosen so that $\int_{-1}^{1} Q_n(x)\,dx = 1$

let $\varepsilon > 0$ be given

then $\exists\ \delta(\varepsilon)$ s.t.

$|x-y| < \delta$

$\Rightarrow |f(x)-f(y)| < \varepsilon$

(f is unif cont on $[0,1]$)

Properties (Rudin's def)

Using $(1-x^2)^n \geq 1-nx^2$

get:

1) $c_n < \sqrt{n}$

2) for any $\delta > 0$, $Q_n(x) \leq \underbrace{\sqrt{n}\,(1-\delta^2)^n}_{\to 0 \text{ as } n\to\infty}$

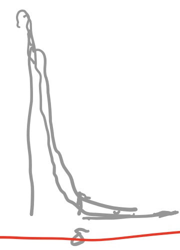
(Rudin ...:

$\frac{1}{2}\log n + n\log(1-\delta^2)$

$\to -\infty$ as $n\to\infty$ and $\log(1-\delta^2)<0$

3) in part, for any fixed $\delta>0$,

$Q_n(x)\to 0$ uniformly on $[\delta,1]$



properties of $Q_n(x)$

0) $Q_n(x)\geq 0$

1) $\forall\,\delta>0$ $Q_n(x)\to 0$ unif on $[\delta,1]$, $[-1,-\delta]$

2) $\int_{-1}^{1} Q_n(t)\,dt = 1$

Let $M = \sup_{t\in[0,1]} |f(x)|$

Given $\varepsilon>0$ $\exists\,\delta>0$ s.t. $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon/2$ $\forall\, x,y\in[0,1]$

$P_n(x) = \int_{-1}^{1} f(x+t)\,Q_n(t)\,dt = \int_{-1}^{1} f(t)\,Q_n(t-x)\,dt$ $P_n(x)$

given $f\in C([0,1],\mathbb{R})$, Given $\varepsilon>0$, want $N=N(\varepsilon)$

4) $|P_n(x) - f(x)| < \varepsilon$

$\int_{-1}^{1} f(x+t)\,Q_n(t)\,dt$

$\leq \int_{-1}^{1} |f(x+t) - f(x)|\,Q_n(t)\,dt$

$f(x) = \int_{-1}^{1} f(x)\,Q_n(t)\,dt$ $f(x)\int_{-1}^{1} Q_n(t)\,dt = f(x)$

$=$

$\left|\int_{-1}^{-\delta}\right| + \int_{-\delta}^{\delta} + \left|\int_{\delta}^{1}\right|$

three different estimates

$\leq \left|\int^{-\delta}\right| + \left|\int_{-\delta}^{\delta}\right| + \left|\int_{\delta}^{1}\right|$

$$\left|\int_{-1}^{\delta} \underbrace{|f(x+t)-f(x)|}_{\le 2M}\ \underbrace{Q_n(t)}_{\text{small}}\,dt\right| \le 2M\int_{-1}^{-\delta} Q_n(t)\,dt$$

$\le |f(x+t)| + |f(x)|$

$\le \sup |\cdots|$

$= M+M$

$= 2M$

$$\le 2M\int_{-1}^{\delta}\boxed{\sqrt{n}(1-\delta^2)^n}\,dt$$

$$\le \boxed{2M\sqrt{n}(1-\delta^2)^n}$$

Same for $\left|\int_{\delta}^{1}\right|$

Now $\quad \underbrace{\left|\int_{-\delta}^{\delta} \underbrace{|f(x+t)-f(x)|}_{\le \varepsilon/2}\,Q_n(t)\,dt\right|}_{\le \frac{\varepsilon}{2}\underbrace{\int_{-\delta}^{\delta}Q_n(t)\,dt}_{\le 1}}$

$< \varepsilon/2$

So sum

$$\le \underbrace{4M\sqrt{n}(1-\delta^2)^n}\ + \varepsilon/2$$

$\exists N$ s.t. $n>N \Rightarrow 4M\sqrt{n}(1-\delta^2)^n < \frac{\varepsilon}{2}$

So sum $\le \varepsilon$ for $n>N$ uniformly in x

$Q_n(x)$: $\begin{cases} \int_{-1}^{1} Q_n(x)\,dx = 1 \\ \forall \delta>0,\ Q_n(x)\to 0 \text{ unif on } [-1,-\delta] \cup [\delta,1] \end{cases}$

"approximate identity"

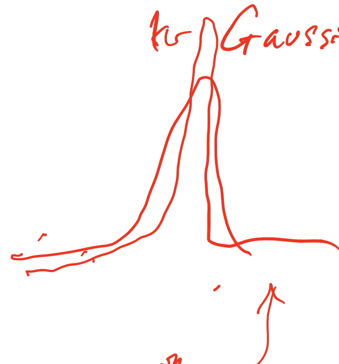
because $\int^{1} f(x+t)\,Q_n(t)\,dt \approx f(x)$

"convolution with ..."

$\frac{1}{\sqrt{2\pi}} e^{-x^2}$

Weirstrass poly approx

to Gaussian



$c_n (1 - x^2)^n$

Many other choices

of poly approx identities

"approximation thry"

Chebyshev poly "best approx"

reduce to

0 1

Example

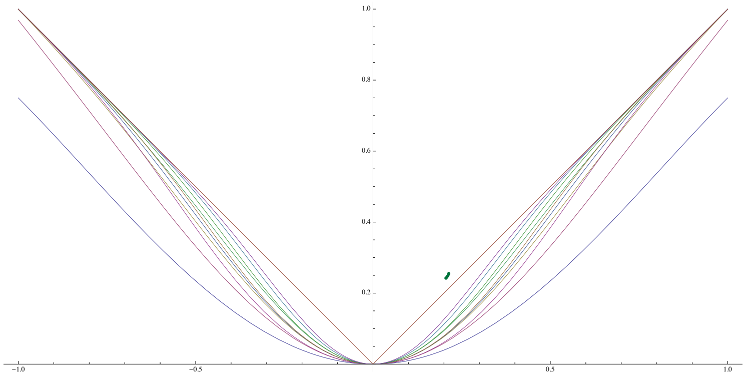
$Q_n(t)$:

$$f \to \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt.$$

Rudin, chapter 7, Exercise 23.

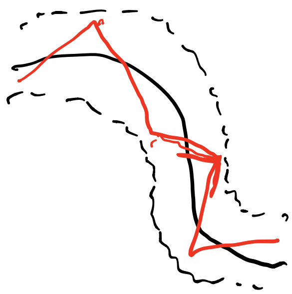
Approximating $|x|$ on $[-1.1]$

Out[83]=

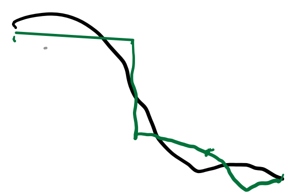


Lebesgue's pf of Weirstrass (from ref above)

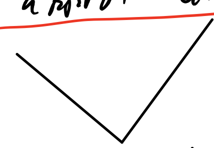
$\epsilon > 0$



1) any cont f can b uniformly approx by piecewise linear functions:



2) if you can approx absolute value



uniformly by polys, then same for piecewise linear.

3) approx $|x|$.

Rmk: not a very good approx

Cor: $C([0,1], \mathbb{R})$ is separable $\|f\|_\infty$

(has a countable dense set)

$\mathbb{Q} \subset \mathbb{R}$ dense, countable

take polys with $\mathbb{Q}$-coefficients

$\mathbb{Q}[x] \subset \mathbb{R}[x]$ dense, countable.

Algebras of Functions

$\mathcal{A} \subset \mathcal{F}(E, \mathbb{R}) = \{$all funcs $f: E \to \mathbb{R}\}$

Define $\mathcal{A}$

- Algebra of functions on a set E.

collec of funcs on E s.t.

$$\begin{cases} 1) & f, g \in \mathcal{A} \Rightarrow f+g \in \mathcal{A} \\ 2) & f, g \in \mathcal{A} \Rightarrow fg \in \mathcal{A} \\ 3) & f \in \mathcal{A},\ c \in \mathbb{R} \Rightarrow cf \in \mathcal{A} \end{cases}$$

Note! not assume that const func $\in \mathcal{A}$.

- $\mathcal{B}$ = uniform closure of $\mathcal{A}$.

$\Leftarrow \{$uniform limits of funcs in $\mathcal{A}\}$

Theorem

If $\mathcal{A}$ is an algebra of $\boxed{\text{bounded}}$ functions, then its uniform closure $\mathcal{B}$ is uniformly closed. and is an algebra.

f, g $\lim (f_n g_n) = (\lim f_n)(\lim g_n)$

OK bounded.

Gk fun

Interest Algebras of Cont funcs on a Compact Space.

$\Rightarrow$ bounded.

Exs of ~~algebra~~
sub-algebras of $C(K,\mathbb{R})$

$\to$ $C([0,1];\mathbb{R})$

{polynomials} a subalgebra.

~~polynomials with ...~~

even polys

even func.

$T \subset C[0,2\pi]$

trigonometric ~~polynomials~~ $\sum_{n=n_0}^{n_1} a_n \cos nx + b_n \sin nx$

$$\sum_{n=n_0}^{n_1} a_n \cos nx + b_n \sin nx$$

$a_i, b_n \in \mathbb{R}$

$\mathbb{C}$-fun. $\mathbb{C}$-alg $\Rightarrow$ $\mathbb{C}$-valued func. scalar $\mathbb{C}$.

$$\sum_{n=n_0}^{n_1} a_n e^{inx} \quad a_n \in \mathbb{C}$$

$$\boxed{e^{imx} e^{inx} = e^{i(m+n)x}}$$

$\boxed{\mathcal{A} \subset C(K, \mathbb{R})}$ K compact

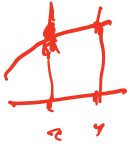
Definitions:

- $\mathcal{A}$ separates points

$\forall\, x, y \in K,\ x \neq y\ \exists\, f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$

- Equivalent formulation:

$\forall\, x \in K\ \exists\, f \in \mathcal{A}$ with $f(x) \neq 0$



- $\mathcal{A}$ vanishes at no point.

$\forall\, x, y \in K,\ x \neq y,\ \forall\, c_1, c_2 \in \mathbb{R}$

$\exists\, f \in \mathcal{A}$ $\quad f(x) = c_1,\ f(y) = c_2.$

Stone's Generalization

Theorem (Stone–Weierstrass thm)

- K compact space and $\mathcal{A}$ = sub-algebra of $\mathcal{C}(K, \mathbb{R})$.
- Suppose $\mathcal{A}$ separates points and vanishes at no point.
- Then the uniform closure of $\mathcal{A}$ is all of $\mathcal{C}(K, \mathbb{R})$

Ex $K = [0,1]$ and P = polynomials

1) P is a subalgebra of $C([0,1], \mathbb{R})$

2) separates points $x \neq y$ $\exists\, p$ s.t. $p(x) \neq p(y)$

(Weierstrass thm $\Leftarrow$

Stone's thm

$\mathcal{F}$ collection of fns $\in C(K,\mathbb{R})$

s.t.

$f, g \in \mathcal{F} \Rightarrow \max\{f,g\}$, $\min\{f,g\}$ both in $\mathcal{F}$

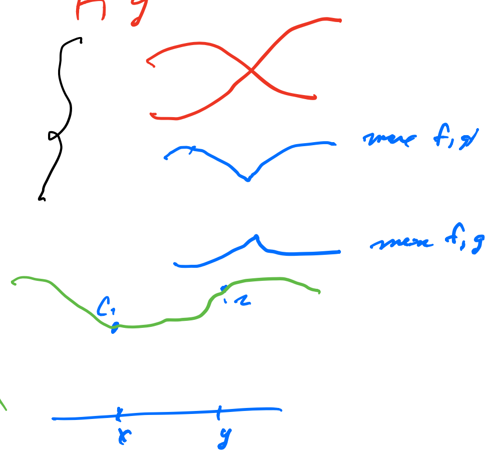
$\forall\, x \neq y \in K$, $x \neq y$

$\forall\, a, b \in \mathbb{R}$

$\exists\, f \in \mathcal{F}$ w/ $f(x) = a$, $f(y) = b$

$\Rightarrow \mathcal{F}$ dense in $C(K,\mathbb{R})$

f, g



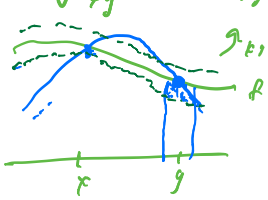
$\Rightarrow$ unif dense in $C(K,\mathbb{R})$

Sketch Pf. given $f \in C(K,\mathbb{R})$, $\varepsilon > 0$

$x, y \in K$ $x \neq y$

$\exists\, g_{xy}$ s.t. $g_{xy}(x) = f(x)$

$g_{xy}(y) = f(y)$



f cont $\Rightarrow \exists$ nbr U_y of y

s.t. $g_{xy}(z) > f(z) - \varepsilon$

$\forall\, z \in U_y$

fix x, move y

$\{U_y\}$ open cover of K

$\Rightarrow \exists\, y_1, \dots, y_n \quad U_{y_1}, \dots, U_{y_n}$ cover K

~~min~~ $g_{xy_1}, \dots, g_{xy_n}$
max

$g_x = \cancel{\min}^{\max}\{g_{xy_1}, \dots, g_{xy_n}\}$

$g_x(z) > \underline{f(z) - \varepsilon} \quad \forall z \in K$

$g_x = \cancel{\max}(g_{xy_1}, \dots, g_{xy_n})$

$> f(z) - \varepsilon$

~~min for each x~~

See correct pf below

Closed under max
min
+ other pr.

given $\varepsilon > 0$ $\exists\, \varphi \in \mathcal{F}$
s.t. $|f(x) - \varphi(x)| < \varepsilon$
$\forall x \in K$

3) $\forall \varepsilon \in [0,1]$ $\exists$ p s.t. $p(x) \neq 0$.

Stone's thm

K compact, $\mathcal{F} \subset C(K,\mathbb{R})$ satisfies:

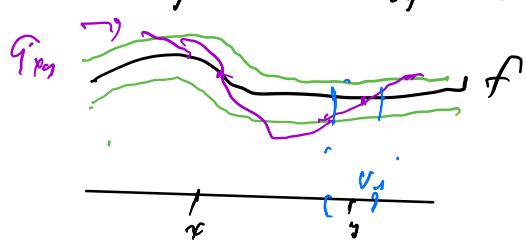
1) $f, g \in \mathcal{F} \Rightarrow \max\{f,g\}, \min\{f,g\} \in \mathcal{F}$

2) $\forall a,b \in \mathbb{R}$, $\forall x \neq y \in K$, $\exists f \in \mathcal{F}$ s.t. $f(x) = a$, $f(y) = b$

$\Rightarrow \bar{\mathcal{F}} = C(K,\mathbb{R})$

Pf: Let $f \in C(K,\mathbb{R})$, and $x, y \in K$, $x \neq y$.

$\exists\, g_{xy} \in \mathcal{F}$ st. $g_{xy}(x) = f(x)$, $g_{xy}(y) = f(y)$



$\forall \varepsilon > 0$ $\exists\, U_y$ st. $z \in U_y$

$\Rightarrow g_{xy}(z) > f(z) - \varepsilon$

Fix x, vary y: $\{U_y\}$ open cover of K

$\Rightarrow \exists\, y_1, \dots, y_n$ st. $U_{y_1}, \dots, U_{y_n}$ cover K

let $h_x = \sup\{g_{xy_1}, \ldots, g_{xy_n}\} \in \mathcal{L}$

$g_{xy_j}(x) < f(x) \Rightarrow \exists$ nbd V_x s.t.

$\Rightarrow h_x(z) < f(z) + \varepsilon$

$\{V_x\}_{x \in K}$ open cover

$\Rightarrow \exists\, x_1, \ldots, x_m$ s.t. $V_{x_1}, \ldots, V_{x_m}$ cover K

Let $h = \inf\{h_{x_1}, \ldots, h_{x_m}\}$

$h \in \mathcal{L}$ & $|f(z) - h(z)| < \varepsilon$

$\forall x \in K$

To prove Stone-Weierstrass

Let $\mathcal{B} = \bar{\mathcal{A}}$

$=$ unif closure of $\mathcal{A}$,

1) $f \in \mathcal{A} \Rightarrow |f| \in \mathcal{B}$

ply $\to |x|$

2) $f, g \in \mathcal{A} \Rightarrow \left. \begin{array}{l} \max\{f,g\}, \\ \min\{f,g\} \end{array} \right\} \in \mathcal{B}$

$\frac{1}{2}$ $f(x)+g(x)+|f(-)$

$|-|$

3) Use Stone's thm.

QED