Foundations of Analysis II
Week 3

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$\boxed{\text{Look at Rudin, chap 8, problems 2-5}}$

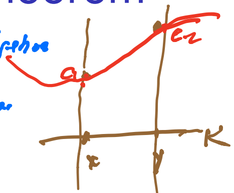
$e^{ix} = \cos x + i \sin x$

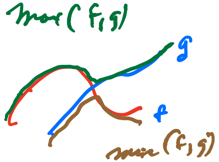
Recall Stone's Theorem

Does not Assume $\mathcal{F}$ is an algebra

how alg ∃ thm can be applied

We'll see Theorem





Let K be compact and let $\mathcal{F} \subset \mathcal{C}(K, \mathbb{R})$ satisfy:

- *For all $x, y \in K$ with $x \neq y$ and for all $c_1, c_2 \in \mathbb{R}$ there exists $f \in \mathcal{F}$ such that $f(x) = c_1$ and $f(y) = c_2$.*
- *If $f, g \in \mathcal{F}$, then $\max\{f, g\}$ and $\min\{f, g\}$ are also in $\mathcal{F}$.*
- *Then $\mathcal{F}$ is uniformly dense in $\mathcal{C}(K, \mathbb{R})$ (that is, $\mathcal{F}$ is dense in $\mathcal{C}(K, \mathbb{R})$ in the ∞-norm)*

uniform closure of $\mathcal{F}$ is $\mathcal{C}(K,\mathbb{R})$

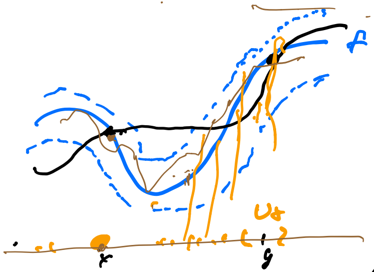
These 2 props satisfied $\mathcal{C}(K,\mathbb{R})$

dense in $\|\cdot\|_\infty$

$\Leftrightarrow$ $\forall f \in \mathcal{C}(K,\mathbb{R})$ & $\forall \varepsilon > 0$ $\exists g \in \mathcal{F}$ s.t. $|f(x) - g(x)| < \varepsilon$ $\forall x \in K$

Proof of Stone's theorem

- Let $f \in \mathcal{C}(K, \mathbb{R})$ and let $\epsilon > 0$ be given.
- For all $x \in K$, there exists $g_x \in \mathcal{F}$ such that $g_x(x) = f(x)$ and $g_x(z) > f(x) - \epsilon$ for all $z \in K$.



$\varepsilon > 0$

(1) $\exists\, g_{xy} \in \mathcal{F}$ s.t. $g_{xy}(x) = f(x)$, $g_{xy}(y) = f(y)$

(2) g_{xy} cont $\Rightarrow \exists$ nbd U_y of y s.t. $g_{xy}(z) > \underline{f(z) - \varepsilon}$ $\forall z \in U_y$

(3) K compact $\Rightarrow$ covered by fin many $U_{y_1}, \ldots, U_{y_m}$

$g_{x,y_1}, \dots, g_{x,y_n}$

$g_{x,y_i}(x) = f(x)$

$g_{x,y_i}(y_i) = f(y_i)$

$g_{x,y_i}(z) > f(z) - \varepsilon$ on U_{y_i}

let $g_x = \max\{\underbrace{g_{x,y_1}, \dots, g_{x,y_n}}\}$

each $> f(z) - \varepsilon$ on ~~some~~ U_{y_i}

$\Rightarrow g_x(z) > f(z) - \varepsilon \quad \forall z \in K.$

- There exist $x_1, \ldots, x_m \in K$ and an open cover $\{V_1, \ldots, V_m\}$ of K such that $x_i \in V_i$ and $g_{x_i}(z) < f(z) + \epsilon$ for all $z \in V_i$.

$g_{x_i}(x_i) = f(x_i)$ & cont $\Rightarrow \exists$ nbd V_{x_i} of x_i

$g_{x_i}(z) < f(z) + \varepsilon \quad \forall z \in V_{x_i}$

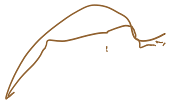
$\Rightarrow$ min

- Let $g = \min\{g_{x_1}, \ldots, g_{x_m}\}$ Then $g \in \mathcal{F}$ and $|f(z) - g(z)| < \epsilon$ for all $z \in K$.

$$f(z)-\varepsilon < g(z) < f(z)+\varepsilon \qquad \forall\, z \in K$$

$\longrightarrow$ unif closure

Lebesg…

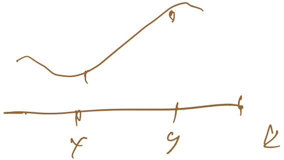


Stone'-Weierstrass Theorem

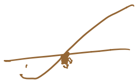
Theorem

- *K compact space and $\mathcal{A} =$ sub-algebra of $\mathcal{C}(K, \mathbb{R})$.*
- *Suppose $\mathcal{A}$ separates points and vanishes at no point.*
- *Then the uniform closure of $\mathcal{A}$ is all of $\mathbb{C}(K, \mathbb{R})$*

Recall



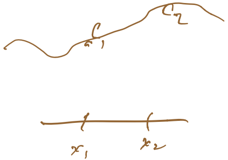
$C(K, \mathbb{R})$ satisfies this



- $\mathcal{A}$ separates points means: for all $x, y \in K, x \neq y$, there exists $f \in \mathcal{A}$ with $f(x) \neq f(y)$.
- $\mathcal{A}$ vanishes at no point of K means: for all $x \in K$ there exists $f \in \mathcal{A}$ with $f(x) \neq 0$.

$\Updownarrow$

- These two conditions are equivalent to the following condition: For all $x_1, x_2 \in K, x_1 \neq x_2$ and for all $c_1, c_2 \in \mathbb{R}$ there exists $f \in \mathcal{A}$ with $f(x_1) = c_1$ and $f(x_2) = c_2$.



$\exists f$ s.t. $f(x_1) \neq f(x_2)$

$\boxed{\begin{aligned} f(x_1) &= c_1 \\ f(x_2) &= c_2 \end{aligned}}$

$c_1 \neq c_2$

$\dfrac{f}{f(1)}, f(x_1)$

Proof of Stone-Weierstrass

$f \in \mathcal{A} \Rightarrow |f| \in \mathcal{B} \Rightarrow f \in \mathcal{B} \Rightarrow |f| \in \mathcal{B}$

- Let $\mathcal{B}$ = uniform closure of $\mathcal{A}$.
- $\boxed{f \in \mathcal{B} \Rightarrow |f| \in \mathcal{B}}$

Observe: If $P(t)$ is a real polyn

$p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$

and $f \in \mathcal{A} \Rightarrow p(f) \in \mathcal{A}$

$p(f)(x) = p(f(x))$

Range f

~~Let P_n = polys that $\to |x|$~~

$f \in \mathcal{A}$, want $|f| \in \mathcal{B}$

that approx $|x|$ on

$\exists C > 0$ st $|f(x)| \le C$ take P_n st $[-C-1, C+1]$

$P_n(f) \in \mathcal{A}$ & $P_n(f) \to |f|$ unif on K.

- $\underbrace{f, g \in \mathcal{B}} \Rightarrow \max\{f, g\}, \min\{f, g\} \in \mathcal{B}$

$f-g \in \mathcal{B} \Rightarrow |f-g| \in \mathcal{B}$ $\Rightarrow$

max $= \frac{1}{2}\left((f+g) + |f-g|\right)$

min $= \frac{1}{2}\left((f+g) - |f-g|\right)$

- Apply Stone's theorem to $\mathcal{B}$.

$\mathcal{B}$ is dense in $\mathcal{C}(K, \mathbb{R})$

& $\mathcal{B}$ -- closed ---

$\mathcal{B} = \mathcal{C}(K, \mathbb{R})$

Stone-Weierstrass for Complex Functions

Definition

scalars $\mathbb{C}$

valued

A $\mathbb{C}$-algebra $\mathcal{A}$ of complex functions on a set E is called *self-adjoint* if and only if it is closed under complex conjugation: $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$.

Theorem

- *K compact space and $\mathcal{A} =$ self-adjoint sub-algebra of $\mathcal{C}(K, \mathbb{C})$.*
- *Suppose $\mathcal{A}$ separates points and vanishes at no point.*
- *Then the uniform closure of $\mathcal{A}$ is all of $\mathcal{C}(K, \mathbb{C})$*

Proof

- Let $\mathcal{A}_{\mathbb{R}}$ be the collection of real functions in $\mathcal{A}$.
- If $f = u + iv \in \mathcal{A}$, u, v real functions on K, then

$$u = \frac{f + \bar{f}}{2} \text{ and } v = \frac{f - \bar{f}}{2i} \text{ are both } \in \mathcal{A}_{\mathbb{R}}$$

- Thus $\mathcal{A} = \{u + iv : u, v \in \mathcal{A}_{\mathbb{R}}\}$
- Apply Stone-Weierstrass to $\mathcal{A}_R$

Applications

- Weierstrass theorem: Let $\mathcal{P} \subset \mathcal{C}([a, b])$ be the sub-algebra of polynomials. The the uniform closure of $\mathcal{P}$ is $\mathcal{C}([a, b])$.

$\mathbb{R}$

$\mathbb{C}$

$x, y \in [a,b]$ $\exists$ p s.t.

$p(x) \neq p(y)$?

$p(x) = x$

$a_0 + a_2 t^2 + a_4 t^4 \cdots$ on $[0,1]$

- Let $\mathcal{P}^{ev}$ be the subalgebra of $\mathcal{P}$ consisting of polynomials with all monomial terms of even degree. Then the uniform closure of $\mathcal{P}^{ev}$ in $\mathcal{C}([0,1])$ is all of $\mathcal{C}([0,1])$.

Vanishes at no pts

1

t $\quad$ t^2 $\quad$ $[0,1]$

$x \neq y$ $\quad$ $x^2 \neq y^2$

$f(x) = x \notin \overline{\mathcal{P}^{ev}}$

-1 $\quad$ 0

- What is the uniform closure of $\mathcal{P}^{ev}$ in $\mathcal{C}([-1,1])$?

$\mathcal{C}([-1,1],\mathbb{R})$

even $f(x)=f(-x)$

$-1/2$ $\quad$ $1/2$

$f(\alpha)^2=\alpha^2$

- Which hypothesis of Stone-Weierstrass fails?

Sep. pts

Chap: power series → e^x, $\sin x$, ...

fourier series

Γ- fun

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \cdots$$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$$\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left[\theta - \theta^3/3! + \theta^5/5! - \cdots\right]$$

Trigonometric Polynomials and Fourier Series

- Let $S^1 =$
  - The unit circle $|z| = 1$ in $\mathbb{C}$
  - $\Leftrightarrow \{e^{i\theta} : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$
  - $\Leftrightarrow \mathbb{R}/2\pi\mathbb{Z}$

  A compact space.
- $\mathcal{C}(S^1)$ is the algebra of continuous functions on $\mathbb{R}$ which are periodic of period 2π
- Let $[a, b] \subset \mathbb{R}$ be any interval of lenth 2π (for example, $[0, 2\pi]$ or $[-\pi, \pi]$). Then $\mathcal{C}(S^1)$ is the subalgebra of $\mathcal{C}([a, b])$ of all f with $f(a) = f(b)$.

$e^{i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta + i\sin\theta$

$\mathbb{R}/2\pi\mathbb{Z}$

$\theta \in \mathbb{R} \mapsto \theta + 2\pi$

$\to \theta + 4\pi$

$\theta - 6\pi$

$\mathbb{R}/2\pi\mathbb{Z}$ = equiv. rel. $x \sim y \iff x - y \in 2\pi\mathbb{Z}$

$\exists n \in \mathbb{Z}$ s.t. $x - y = 2\pi n$

$e^{i\theta}$

$\theta \quad \mathbb{R}/2\pi\mathbb{Z}$

- Let $\mathcal{A} \subset \mathcal{C}(S^1, \mathbb{C})$ be the subalgebra of functions

$$\boxed{f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}}$$

$\theta \in \mathbb{R}$

$c_n \in \mathbb{C}$

for some N

where the c_n are complex constants, $N = 0, 1, 2, \ldots$

- The elements of $\mathcal{A}$ are called *trigonometric* "*polynomials*"

negative powers

$e^{in\theta} = (e^{i\theta})^n$

$$\overline{\left(\sum_{-N}^{N} c_n e^{in\theta}\right)} = \sum_{-N}^{N} \bar{c}_n e^{-in\theta}$$

A trigon poly of degree N

$$f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$$

$c_n \in \mathbb{C}$

$N=0 \quad c_0$

$N=1 \quad c_{-1}e^{-i\theta} + c_0 + c_1 e^{i\theta}$

$N=2 \quad c_{-2}e^{-2i\theta} + c_{-1}e^{-i\theta} + c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta}$

$\vdots$

$e^{i\theta}$ easier than $\cos\theta, \sin\theta$

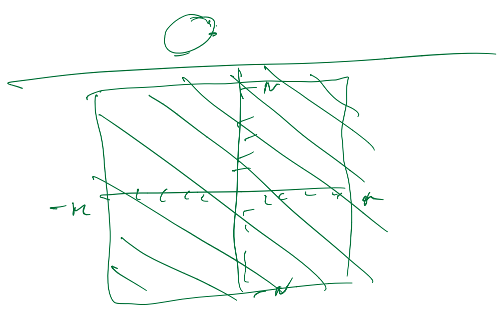
symmetric

$-N \quad 0 \quad N$

Closed under complex conj

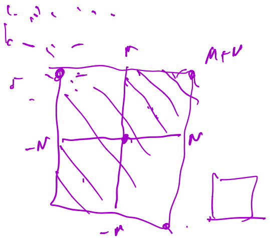
(self-adjoint) $\mathbb{C}$-algebra of $\mathbb{C}$-fns on $C(K,\mathbb{C})$

$e^{i\theta}$ separates



$(e^{i\theta})^2 = e^{2i\theta}$

$c_{-N} e^{-iN\theta} + c_{-(N-1)} e^{i(-N+1)\theta} + \dots + c_{-1} e^{-i\theta} + c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} \dots c_N e^{iN\theta}$



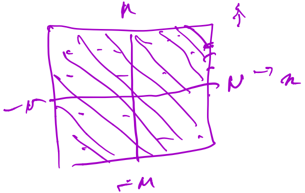
- Check that $\mathcal{A} \subset \mathcal{C}(S^1, \mathbb{C})$ is a self-adjoint algebra that separates points and does not vanish at any point.
- Stone-Weierstrass $\Rightarrow$ the uniform closure of $\mathcal{A}$ is $\mathcal{C}(S^1, \mathbb{C})$
- Any continuous $\mathbb{C}$-valued periodic function on $\mathbb{R}$ with period 2π can be uniformly approximated by trigonometric polynomials.

$$\left(\sum_{-N}^{N} c_n e^{in\theta}\right)\left(\sum_{-M}^{M} a_\ell e^{i\ell\theta}\right)$$

$$\sum \sum_{\substack{-N \le n \le N \\ -M \le \ell \le M}} c_n a_\ell e^{i(n+\ell)\theta}$$

$$= \sum \left(\sum_{m+l=k} c_m d_l \right) e^{i(m+l)\theta}$$

$-(M+N) \le k \le M+N$



$$\sum_{-N}^{N} c_n e^{in\theta}$$

$$e^{in\theta} = \cos n\theta + i \sin n\theta$$

$c_n = (a_n - i b_n)$ $a_n, b_n \in \mathbb{R}$

d

$$\sum_{-N}^{N} (a_n - i b_n)(\cos n\theta + i \sin n\theta)$$

$$= \sum_{-N}^{N} (a_n \cos n\theta + b_n \sin n\theta) + i(\text{------} \cdot)$$

a_n $\sum$ ...

zero for $n = 0, 1, \ldots$

$$\boxed{a_0 + \sum_{n=1}^{N} (a_n \cos n\theta + b_n \sin n\theta)}$$

Real Trigonometric Polynomials

cx trig poly $\sum_{n=-N}^{N} c_n e^{in\theta}$

$P(x)$

$$\overline{P(x)} = \sum_{-N}^{N} \overline{c_n e^{in\theta}}$$

$$= \sum_{-N}^{N} \overline{c_n} e^{-in\theta}$$

$\overline{e^{ix}}$ $x \in \mathbb{R}$

$= e^{-ix}$ $x \in \mathbb{R}$

pf. $e^{ix} = \cos x + i \sin x$

$$\overline{e^{ix}} = \cos x - i\sin x = \cos(-x) + i\sin(-x)$$

$$= e^{i(-x)} = e^{-ix}$$

Fourier: Solving ~~partial dif~~ PDE's

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

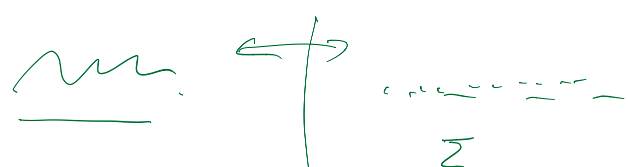
$$\frac{d}{d\theta}\left(\sum_{-\infty}^{\infty} c_n e^{in\theta}\right)$$

"hope" "=" $\displaystyle\sum_{-\infty}^{\infty} \frac{d}{d\theta}\left(c_n e^{in\theta}\right)$

$$= \sum_{-\infty}^{\infty} in\, c_n e^{in\theta}$$

$$\frac{d}{d\theta} \longleftrightarrow in$$

$$f(\theta) \to \frac{df}{d\theta} \longleftrightarrow \{c_n\}_{n=-\infty}^{\infty} \to \{in\, c_n\}$$



$\frac{d}{d\theta}$

$C([0,1]) \xrightarrow{I} C(\ )$

$\subseteq$ $\quad D(\ )$

$D(\ ) \xrightarrow{D} C(\ )$

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta} \quad \text{converge?}$$

Simple sufficient condi

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty \implies \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

converg uniformly.

$\{c_n\}$

$\underline{\text{Def}}$ $\ell^p = \{ \{c_n\}_{-\infty}^{\infty} : \sum |c_n|^p < \infty \}$

just proved:

$\ell^1 \longrightarrow C_0(S^1, \mathbb{C})$ unit circle $\{e^{i\theta}\}$

$\{c_n\} \longrightarrow \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$

$\theta \in \mathbb{R}/2\pi\mathbb{Z}$

$\mathbb{R}/2\pi\mathbb{Z}$

$[0, 2\pi]$

$[-\pi, \pi]$

$C(S^1) = \{ f \in C(\mathbb{R}) : f(x+2\pi) = f(x) \ \forall x \}$

$\{c_n\} \in \ell^1 \longrightarrow \sum c_n e^{in\theta}$ cont per fn on $\mathbb{R}$

$\sum |c_n| < \infty$

diff ?

$\frac{d}{d\theta}\left(\sum c_n e^{in\theta}\right) = \sum in\, c_n e^{in\theta}$

$= \frac{df}{d\theta}$ cont

$\Rightarrow \sum |n c_n| < \infty$

$\{c_n = 1/n^2\}_{\ell^1}$ $\sum \frac{1}{n^2} e^{in\theta}$

$n c_n = 1/n$ $\sum |1/n| = \infty$ not ℓ^1

$\underline{\text{Heat equation}}$ "Théorie Analytique de la Chaleur"

$\{c_n\} \to n c_n$

$c_n \to c_n/n$ $n \neq 0$

Fourier Series

$|e^{in\theta}| = 1$

- Infinite series

$$\left| f(\theta) = \sum_{-\infty}^{\infty} c_n\, e^{in\theta} \right| \le \sum |c_n|$$

$$\sum |c_n| < \infty$$

- Convergence?
- Norms $1, 2, \infty$? Which?

$\ell^1 < \infty \implies \|f\|_\infty < \infty$

$$\implies \sum_{-\infty}^{\infty} c_n e^{in\theta}$$

[converges]

$\sum |c_n| \cdots$

$$\langle e^{im\theta}, e^{in\theta}\rangle = \int_0^{2\pi} e^{im\theta}\, \overline{e^{-in\theta}}\, d\theta$$

$$= \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta$$

$$= \begin{cases} \dfrac{e^{i(m-n)\theta}}{m-n}\Big|_0^{2\pi} & m \neq n \\ \int_0^{2\pi} 1\, d\theta & m = n \end{cases}$$

$$\langle x, \bar{x} \rangle = \sum |x_i|^2$$

$x, \bar{x} \in \mathbb{R}$

$x_i^2 \in \mathbb{C}$

$\int fg\, dx$ = complex number

$x_1, \dots, x_2$

$x_1 y_1 + \dots + x_2 y_2 = (x, y)$

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}$$

$$\left\{ \frac{e^{im\theta}}{\sqrt{2\pi}} \right\} \text{ form O-N system}$$

ortho - normal

$\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ fns on $[a,b]$

is O N iff $\int_a^b \varphi_m(x) \overline{\varphi_n(x)}\, dx$

ortho $= \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$ normal

$\{e^{in\theta}\}$ orth not norm

$\frac{e^{in\theta}}{\sqrt{2\pi}}$ ON

ON Systems

inner product spaces

$(\cdot , \cdot)$

- $[a, b]$ an interval, L^2 inner product

$$(f, g) = \int_a^b f(x)\overline{g(x)}dx$$

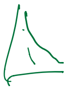
- Makes sense on complex functions satisfying

$$\boxed{\int_a^b |f(x)|^2 dx < \infty}$$

Called *square-integrable functions*, or *functions of class L^2*

ex $f(x) = 1/x$ on $[0,1]$ not in L^2

but $x^{-\alpha}$ $0<\alpha<1/2$



$$\int_0^1 (x^{-\alpha})^2 \, dx < \infty$$

$-2\alpha+1 > 0$

$x^{-1/3}$

$$\int x^{-2\alpha} = \frac{x^{-2\alpha+1}}{-2\alpha+1}\Big|_0^1$$

$x^{-2\alpha+1} \to 0$ as $x \to 0$

$f, g \in L^2$

$\Rightarrow |fg| \in L^1$

- Reason: Schwarz inequality

$$\boxed{\left|\int_a^b f(x)\overline{g(x)}dx\right|^2 \leq \left(\int_a^b |f(x)|^2 dx\right)\left(\int_a^b |g(x)|^2 dx\right)}$$

- Call this space $L^2[a, b]$.
- It is a complex inner product space, just as $\mathbb{C}^n$.
- Think first of $\mathbb{R}^n$, length, angles, etc.

- If $\{\phi_n\}$ is an ON system in $L^2[a,b]$, and

$$\boxed{f = \sum_{-\infty}^{\infty} c_n \, \phi_n}$$

recover the c_n from f by

$$c_n = \int_a^b f(x)\overline{\phi_n(x)}dx$$

$$\frac{e^{inx}}{\sqrt{2\pi}}$$

- c_n called the *Fourier coefficients* of f.
- Write

$$f \sim \sum c_n \, \phi_n$$

$\phi_n(x) \quad e^{inx}$

$$\int \left(\sum_{n=1}^{N} c_n \varphi_n \right) \overline{\varphi_m(x)}\, dx \qquad m \text{ fixed}$$

$$n = 1, \dots, N$$

$$= \sum_{n=1}^{N} c_n \int \varphi_n(x) \overline{\varphi_m(x)}\, dx$$

$= 0$ if $m \neq n$
$= 1$ if $m = n$ $\Big\}$ $\delta_{m,n}$

$$= \sum c_n \delta_{m,n} = c_m$$

$$f = \langle\langle \varphi_1, \dots, \varphi_N \rangle\rangle = \text{span of } \varphi_1, \dots, \varphi_n \qquad L^2$$

$$(f, \varphi_k) \qquad f = \sum c_n \varphi_n$$

$$c_n = (f, \varphi_n)$$

$f, \{\varphi_n\}_{n=1}^{\infty} \to c_n$

$$\boxed{f \approx \sum_{n=1}^{\infty} c_n \varphi_n}$$

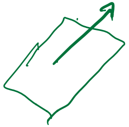
- To study convergence, first finite sums

$$s_N = s_N(f, x) = \sum_{1}^{N} c_n\, \phi_n(x)$$

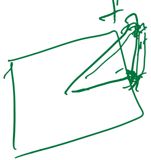
- Minimum property:
  s_N is the vector in span of $\phi_1, \ldots, \phi_n$ closest to f

- Same: s_N is the orthogonal projection of f on the span of $\phi_1, \ldots \phi_n$

$f,\quad c_n$

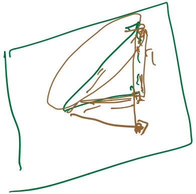


$\langle\langle \varphi_1, \ldots, \varphi_N \rangle\rangle$



$S_N = \perp$: proj of f

on $\langle\langle \varphi_1, \ldots, \varphi_N \rangle\rangle$



$\|S_N\|^2 \le \|f\|^2$

= elmt of $\langle\langle \varphi_1, \ldots, \varphi_N \rangle\rangle$ closest to f

$$\boxed{\sum_{n=1}^{N} C_n^2 \le \int_a^b |f|^2\,dx}$$

$N \to \infty$

$$\boxed{\underline{\sum |c_n|^2} \le \int_a^b |f|^2\,dx}$$

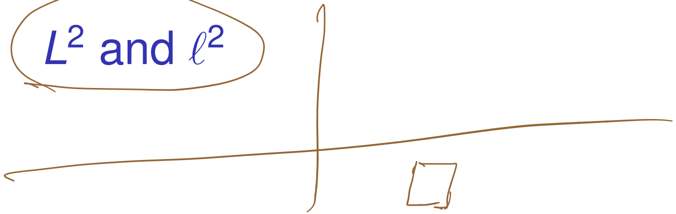
$$\begin{array}{ccc} f \in L^2[a,b] & \longrightarrow & \cancel{\{c_n\}} \\ \big\downarrow & & \hat{f}(n) \\ \{\hat{f}(n)\}_{n \in \mathbb{Z}}^{\infty} & & \end{array}$$

$$f \in L^2(a,b) \Rightarrow (\hat{f}(n)) \in \ell^2$$

For usual Fourier series

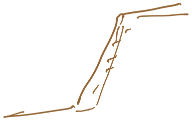
$$\underbrace{L^2[a,b] \xrightarrow{\cong} \ell^2}_{\text{isometry}}$$

L^2 and ℓ^2



$\| \quad \|_1$

$C, \ \|-\|_1$ not conv.



f_n in $C([0,1])$

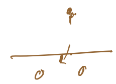
each in $\|--\|_1$

not unif cont

$\|f\|_\infty = 0$

$\Rightarrow f = 0$

$\|f\|_1 = 0 \not\Rightarrow f = 0$

ex:  f/ a.e. zero rel.

$f_n \to f$ discont

$\|f_n - g\|_1 \to 0$, g cont; is this possible?

$\int |f| \, dx = 0$

$\Rightarrow$

f: cont

$\|f - g\|$; h cont on $[0, \tfrac{1}{2}]$, on $(\tfrac{1}{2}, 1]$, $h \geq 0$

$f = g$ a.e.

h = 0 on $[0, \tfrac{1}{2})$ and $[\tfrac{1}{2}, 1]$

$\Rightarrow$ f = g on $[0, \tfrac{1}{2})$, $(\tfrac{1}{2}, 1]$ $\Rightarrow$

and g cont

$\Rightarrow$ g cont on $[0, 1]$,

$$g(x) = \begin{cases} 0 & \text{on } [0, \tfrac{1}{2}) \\ 1 & \text{on } [\tfrac{1}{2}, 1] \end{cases} \quad \begin{matrix} \Rightarrow g(\tfrac{1}{2}) = 0 \\ \Rightarrow g(\tfrac{1}{2}) = 1 \end{matrix}$$

impossible