Foundations of Analysis II
Week 4

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First 15 terms of $\sum \frac{1}{1+n^2 x}$ on $[-0.08, 0.02]$

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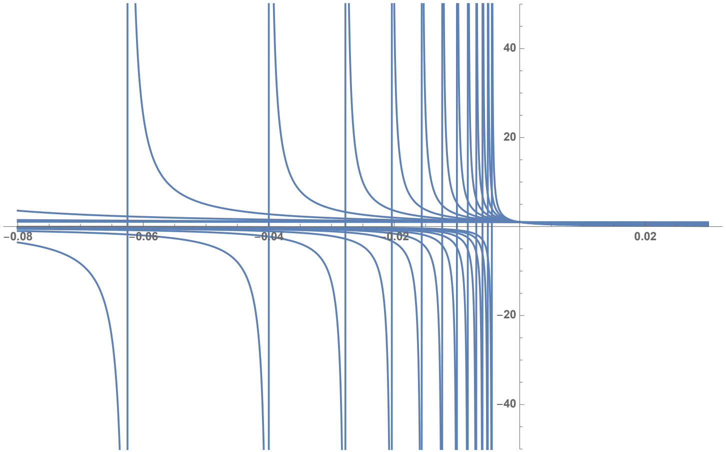
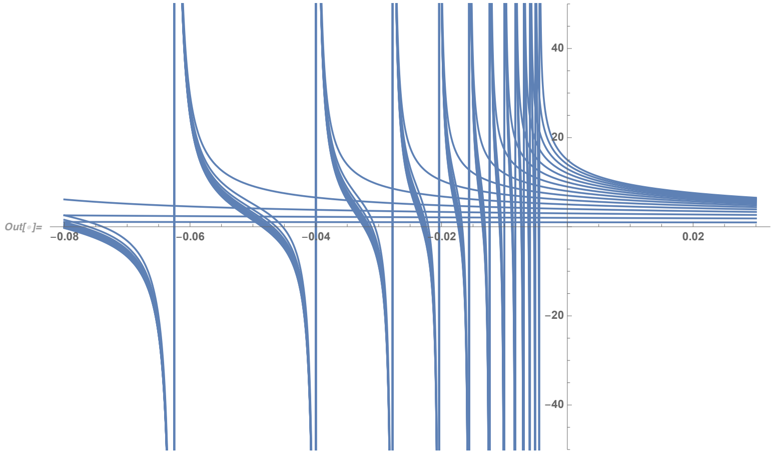


$\sum_{n=1}^{15} \frac{1}{1+n^2x}$ on $[-0.08, 0.02]$



Power Series

- Recall Root Test for $\sum_{n=1}^{\infty} a_n$

- Comparison with geometric series

$\overline{\lim}\, |a_n|^{1/n} < 1 \iff \forall r > \overline{\lim}\, |a_n|^{1/n}$

$|a_n|^{1/n} > r$ for finitely many n.

$\Downarrow$

$\forall r < 1 \ \exists N$ s.t. $n \geq N \implies$

$|a_n|^{1/n} < r \iff |a_n| < r^n$

Compare $\sum r^n \to \frac{1}{1-r}$ if $0 \leq r < 1$.

centered at 0

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

centered at x_0

$(x - x_0)^n$

$x_0 = 0$

- Apply root test to $\boxed{\sum_{n=0}^{\infty} a_n x^n}$
- Radius of convergence

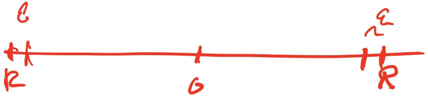
$$R = \boxed{\frac{1}{\limsup(|a_n|^{\frac{1}{n}})}}$$

$$\overline{\lim}\left(|a_n x^n|^{1/n}\right) = \underbrace{\overline{\lim}\, |a_n|^{1/n}\, |x|}_{} < 1$$

$= 1$

> 1

$$|x| < \boxed{\frac{1}{\overline{\lim}\, |a_n|^{1/n}}} = R$$



abs convergence if $|x| < R$

divergence if $|x| > R$ $\quad$ $|x| = R$??

- For every $\epsilon > 0$ uniform and absolute convergence on $\boxed{[-R+\epsilon, R-\epsilon]}$

$R > 0$, $\sum a_n x^n = f(x)$

defines a function on $|x| < R$

- If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

de fine s diff func on $|x| < R$

&

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$\{g_n\}$ $g_n' \to h$, $g_n(x_0)$ conv

$g_n \to g$ and $g_n' \to g'$

diff

R for f $|a_n|^{1/n}$

Same (for f'

$\overline{\lim}\ (n a_n)^{1/n}$ $n^{1/n} |a_n|^{1/n}$

$= \overline{\lim}\ |a_n|^{1/n}$

$\sum n a_n x^n \to$ unif on $[R-\epsilon, R+\epsilon]$

$\Rightarrow f$ is diff and $f' = \sum_{n=1}^{\infty} n a_n x^{n}$

- Iterate: Taylor series.

$f(x) = \sum a_n x^n$

$f'(x) = \sum n a_n x^{n-1}$

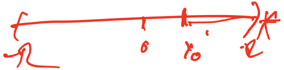
$f'' = \sum n(n-1) a_n x^{n-2}$

$\vdots$

$a_n = \dfrac{f^{(n)}(0)}{n!}$ - - .

- Taylor series of $f(x) = \sum_{n=0}^{\infty} a_n x^n$ at $x_0 \in (-R, R)$

f has a power series representation centered at x_0



$$X = (x - x_0) + x_0$$

$$|(x - x_0) + x_0| < R$$

$$\Uparrow$$

$$|x - x_0| < R - |x_0|$$

$$\sum a_n x^n = \sum a_n \left((x-x_0) + x_0\right)^n$$

$\sum a_n \left((x-x_0)^n + n(x-x_0)^{n-1} x_0 + \frac{n(n-1)}{2}(x-x_0)^{n-2} x_0^2 + \right)$

rearrange $\sum b_m (x-x_0)^m =$

$m \quad 1, \ldots, m$

$a_m (x-x_0)^m + a_m (x-x_0)^{m-1} x_0 + a_{m-1} (x-x_0)^{m-1} \ldots$

Theorem

Suppose $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergenve $R > 0$ and suppose that

$$\{x \in (-R, R) \ : \ \sum_0^\infty a_n x^n = 0\}$$

has a limit point in $(-R.R)$. Then $a_n = 0$ for all n.

$f \equiv 0$

fil

" the zeros of f are isolated,"

~~or the~~

Pf. Suppose $x_0 \in (-R, R)$ is a ~~limit pt of ze.~~ $f(x_0)=0$

Let $f(x) = \sum_{m=0}^{\infty} b_m (x-x_0)^m$ expan at x_0

~~Suppose~~ Suppose not all $b_m = 0$

$\exists$ smallest $b_m \neq 0$

b_{m_0}

$b_{m_0}(x-x_0)^{m_0} + \text{higher}$

$= (x-x_0)^{m}(b_{m_0} + \cdots)$

$= (x-x_0)^{m_0} g(x)$ $\quad g(x_0) \neq 0$

$f(x_0)=0$

$\boxed{f(x) = (x-x_0)^{n_0}\, g(x)}$

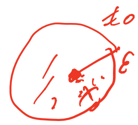
$g(x_0)\neq 0$

$\neq 0$ for $x \neq x_0$ $\qquad \neq 0 \ \forall\ |x-x_0|<\varepsilon$

$\exists\ \varepsilon>0$ s.t. $f(x)\neq 0$ if $0<|x-x_0|<\varepsilon$

$\neq 0$

ε



$\Rightarrow x_0$ not a lim pt of $\{x : f(x)=0\}$

if some $a_n \neq 0 \Rightarrow$ zeros have no limit pt

Contrapos.

all $a_n=0$ $\Leftarrow$ limit pt of zeros

<u>Use:</u> f, g are rep by a power series in $(-R,R)$

$\{x : f(x)=g(x)\}$ has a lim pt. in $(-R,R)$

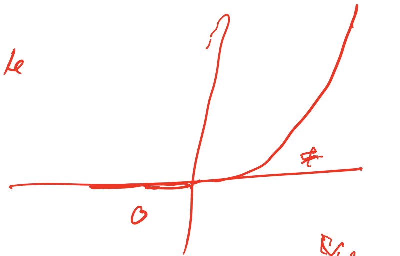
$$\Rightarrow f \equiv g \text{ on } (-R,R)$$

<u>Defn</u> functions rep by power series are called (real) <u>analytic</u>

real analytic $\Rightarrow$ ∞ many derivs $\underline{\underline{C^\infty}}$

$\not\Leftarrow$

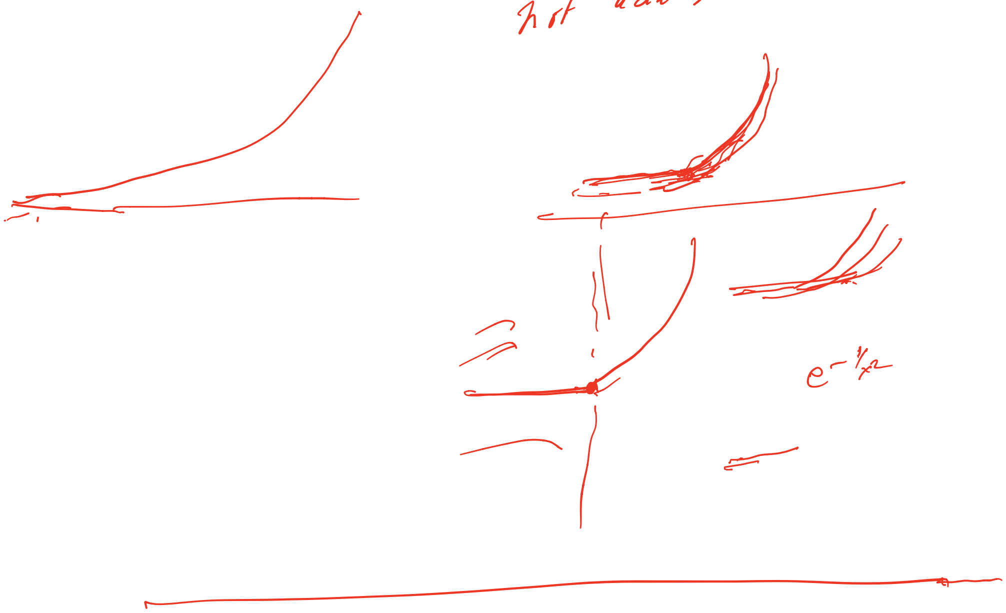
Example



$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

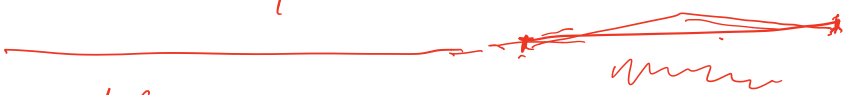
is C^∞

not analytic.

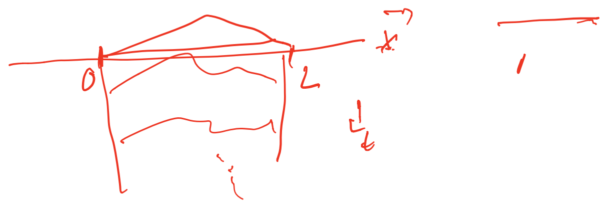


"Vibrating string"

"wave equation"



diff



$u(x,t)$ = height that position, time t.

$u(0,t) = u(L,t) = 0 \quad \forall t$

$u(x,0)$ = given

$u_t(x,0)$ = given

$$u_{tt} = c^2 u_{xx}$$ "wave eq"

Separation of var:

try $u(x,t) = X(x)T(t)$

$u_{xx} = X''(x)\,T(t)$

$u_{tt} = X(x)\,T''(t)$

$u_{tt} = c^2 u_{xx}$

$X T'' = c^2 X'' T$

$$\frac{T''}{T} = c^2 \frac{X''}{X}$$

funct {x,y}

= $\Rightarrow$ constant

d

$$\frac{T''}{T} = d = c^2 \frac{X''}{X}$$

$$\boxed{T'' = d\,T}$$

$$X'' = d\,c^2 X$$

e, ~~$\cosh t$, $\sinh t$, e^{t}~~

$\cosh(\sqrt{d}\,t)$, $\sinh(\sqrt{d}\,\dots$

$-\cancel{\cos}\sqrt{d}\,t$, $\dots$

$d > 0$

$\cosh(\sqrt{d}\,t)$

$\sinh(\sqrt{d}\,t)$

$d < 0$ $\left(\cos(-\sqrt{d}\,t), \sin(-\sqrt{d}\,t)\right)$

$u(0,t) = 0$
$u(L,0) = 0$ > Impose

$\sin\left(\frac{\pi t}{L}\right)$

$$\boxed{\sum \sin\left(\frac{\pi k t}{L}\right) \cos(\dots)}$$

Fourier series

Back to Fourier Series

- Recall $L^2[a,b]$
- Space of complex functions on $[a,b]$ with

$$\int |f(x)|^2 dx, < \infty$$

  (Eventually need Lebesgue integral)
- Inner product

$$(f,g) = \int_a^b f(x)\overline{g(x)}dx$$

- Recall ON system $\{\phi_n\}$ on $[a, b]$:

$$\int_a^b \phi_m(x)\overline{\phi_n(x)}dx = \delta_{m,n} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

- If $f \in L^2[a, b]$ can associate a "Fourier series"

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

where

$$c_n = \int_a^b f(x)\overline{\phi_n(x)}dx$$

$[a,b]$

$f: [a,b] \to \mathbb{C}$

funcs

$$\int_a^b |f(x)|^2 \, dx < \infty$$

" $L^2[a,b]$ "

To be precise, need the Lebesgue integral

For time being, Riemann integral

allow improper

Rmk: the completion of $C[a,b]$

in $\|f\|_1 = \int_a^b |f(x)| \, dx$

= Lebesgue integrable funcs

(larger than R-integrable)

$L^2[a,b]$ $\quad$ $\mathbb{C}$-vector space

"inner product"

$$(f,g) = \int_a^b f(x) \overline{g(x)} \, dx$$

in $\mathbb{C}$. $z, w \in \mathbb{C}$

$z\bar{w}$

$z\bar{z} = \|z\|^2$

$|z|^2$

$(z_1, z_2) \in \mathbb{C}^2$

$(w_1, w_2) \in \mathbb{C}^2$

$z_1\bar{w}_1 + z_2\bar{w}_2$

$z_1\bar{z}_1 + z_2\bar{z}_2 = |z_1|^2 + |z_2|^2$

$= \|(z_1, z_2)\|^2$

$\mathbb{R}^n$

$(x_1, \dots, x_n), (y_1, \dots, y_n)$

$= x_1 y_1 + \dots + x_n y_n$

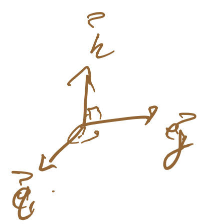
ON $\quad (v_i, v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$(v, w) = 0 \iff v \perp w$

$(v, v) = 1 \iff \|v\| = 1$

$\quad \|v\|^2$

in $\mathbb{R}^3$



ON system

$\mathbb{R}^n \quad e_1 = (1, 0, \dots, 0)$

$e_2 = (0, 1, 0, \dots)$

$\vdots$

e_n

ON system

norm ; lengths

inner prod ; lengths & angles

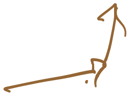
$\|v\|^2 = (v, v)$

"Linear Algebra" and "Euclidean Geometry" give

- Let
$$s_n(f) = s_N(f, x) = \sum_{n=1}^{N} c_n \phi_n(x)$$
be the N^{th} partial sum of the Fourier series, and let
$$< \phi_1, \ldots, \phi_N >$$
denote the span of $\phi_1, \ldots, \phi_N$ in $L^2[a, b]$
- Then $s_N(f)$ is the vector in $< \phi_1, \ldots, \phi_n >$ closest to f.

$v \in \mathbb{R}^3$ $\qquad$ v_1, v_2, v_3 $\quad$ ON

v_1, v_2



$\left(v - (a_1 v_1 + a_2 v_2)\right) \perp v_1 \ \& \ v_2$

$\left(v - (a_1 v_1 + a_2 v_2)\right) \cdot v_1 = 0$

" $\qquad$ " $\qquad$ " $\qquad v_2$

$\cancel{\ }$ $v \cdot v_1 - a_1 (v_1 \cdot v_1) - a_2 (v_2 \cdot v_1) = 0$

$v \cdot v_2 - a_1 (v_1 \cdot v_2) - a_2 (v_2 \cdot v_2) = 0$

$v \cdot v_1 = a_1 \quad v \cdot v_2 = a_2$

$v \quad (v \cdot v_1) v_1 + (v \cdot v_2) v_2$

" = proj of v on $\langle v_1, v_2 \rangle$ = span of v_1, v_2

$(\, , \,) \quad \{v_i\}$

$\langle v_1, \ldots, v_n \rangle$

---

for funcs ON system $\{\varphi_n\}$

$$(\varphi_m, \varphi_n) = \int_{[a,b]} \varphi_n(x) \overline{\varphi_m(x)}\, dx = \delta_{m,n}$$

$$f \sim \sum c_n \varphi_n$$

$$c_n = \int_a^b f(x) \overline{\varphi_n(x)}\, dx$$

$\langle \varphi_1, \ldots, \varphi_N \rangle \quad \sum_{n=1}^{N} c_n \varphi_n$ = vector

in $\langle \varphi_1, \ldots, \varphi_N \rangle$ closest

to f.

$f \mapsto S_N(f) = \sum c_n \varphi_n \qquad c_n = (f, \varphi_n)$

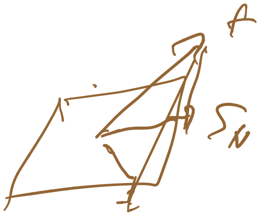
$$= \int_a^b f(x) \overline{\varphi_n(x)}\, dx$$

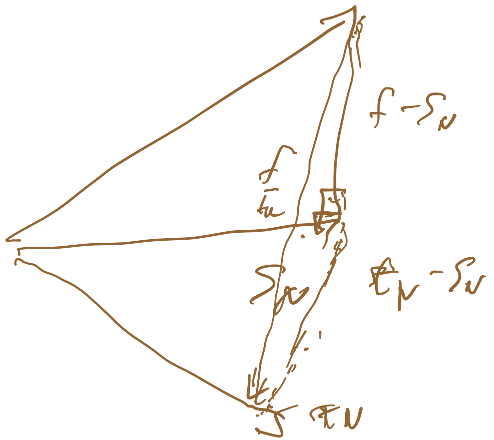
Equivalent formulations:

- If $t_N \in < \phi_1, \ldots, \phi_N >$, then $||f - t_n||^2 \geq ||f - s_N(f)||^2$
- $f - s_N(f) \quad \perp \quad < \phi_1, \ldots, \phi_N >$

$$||f - \sum \gamma_n \varphi_n||^2$$

$$||f - t_N||^2 =$$





f - S_N
f
E_N - S_N
S_N
E_N

$$\|f - S_N\|^2 \le \|f - t_N\|^2$$

$$\|f - t_N\|^2 = \|f - S_N\|^2 + \|t_N - S_N\|^2$$

$$(f - t_N, f - t_N)^2.$$

$$f - t_N = f - S_N + S_N - t_N$$

$$(f - t_N, f - t_N) = ((f - S_N) + (S_N - t_N), (f - S_N) + (S_N - t_N))$$

$$= \|f - S_N\|^2 + \|S_N - t_N\|^2 + 2\underbrace{(f - S_N, S_N - t_N)}_{=0}$$

$f - S_N$ $\qquad$ $f - S_N \perp \langle \varphi_1, \dots, \varphi_N \rangle$

$S_N \qquad N \to \infty$

$$\|S_N\|^2 \le \|f\|^2$$

$$\sum |c_n|^2 \le \|f\|^2$$

They imply Bessel's inequality

- $$\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2 dx$$

$\left(\int |f|^2\right)^{1/2}$

$\int |f|$

- So $f \in L^2[a,b] \Rightarrow \{c_n\} \in \ell^2$

- Ideal situation: this correspondence is an *isometry* between $L^2[a,b]$ and ℓ^2.

- This is the case for usual Fourier series, see Thm. 8.16 in Rudin.

$L^2_g[a,b]$ $\qquad$ ℓ^2, $\{e_1, e_2, \dots\}$

$\sum |c_n|^2 < \infty$

space functions $\xrightarrow{\sim}$ space of sequences

Convergence of Fourier series

$\|\cdot\|_2$

$\sum c_n e^{int} \to f$ in $L^2[-\pi,\pi]$

$f \in$

$L^2(-\pi,\pi) \longleftrightarrow \{c_n\}_{-\infty}^{\infty}$

$\sum |c_n|^2 < \infty$

$$\left(\int |f|^2 dx\right)^{1/2} \qquad \left(\int |f|\right)$$

$$\int_a^b |f|\,dx = \int_a^b |f|\cdot 1\,dx$$

$$\le \left(\int_a^b |f|^2\,dx\right)^{1/2} \left(\int_a^b 1^2\,dx\right)^{1/2}$$

$$= \|f\|_2 (b-a)^{1/2}$$

$$\|f\|_1 \le \sqrt{b-a}\,\|f\|_\infty$$

$$\sum |c_n|^2$$?

$$\sum |c_n| \ldots$$

Trigonometric Series

Usual normalization for trigonometric series:

- $$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$$
- where
$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx} dx$$
- This formula for c_n is correct because
$$\int_{-\pi}^{\pi} e^{imx} e^{-inx} dx = 2\pi\, \delta_{m,n} = \begin{cases} 0 & \text{if } m \neq n, \\ 2\pi & \text{if } m = n. \end{cases}$$

- Observe that $\int_0^{2\pi}$ or $\int_a^{a+2\pi}$ for any a would work as well.
- The associated ON system is

$$\{\frac{e^{inx}}{\sqrt{2\pi}}\}$$

but it's convenient to use the e^{inx} instead.

$$\frac{e^{inx}}{\sqrt{2\pi}}$$

$$e^{inx}$$

$$c_n = \frac{1}{2\pi}\int f(x)\, e^{-inx}$$

L^2 and ℓ^2

- Would like isomorphism.
- Would like to understand other forms of convergence.

Real Trigonometric Series

- If f is real. then

$$\overline{c_n} = \overline{\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx} = \int_{-\pi}^{\pi} f(x)e^{-i(-n)x}\,dx = c_{-n}$$

- So combining n and $-n$ terms in $\sum_{n=-N}^{N} c_n e^{inx}$ get

$$c_0 + \sum_{n=1}^{N}(c_n e^{inx} + c_{-n}e^{i(-n)x}) = c_0 + \sum_{n=1}^{N}(c_n e^{inx} + \overline{c_n e^{inx}})$$

- Let $a_0 = c_0 \in \mathbb{R}$.
- For $n = 1, \ldots, N$, let $a_n, b_n \in \mathbb{R}$ be defined by

$$2c_n = a_n - ib_n.$$

- Then

$$\sum_{-N}^{N} c_n e^{inx} = \boxed{a_0 + \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx))}$$

$\frac{a_0}{2}$

- e^{icmx} mod 1

- $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$
- for $n > 0$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$
- and
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Dirichlet Kernel

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$

- How to sum $s_N(f,x) = \sum_{-N}^{N} c_n e^{inx}$
- Put in definition of c_n and rewrite

$$s_N(f,x) = \sum_{-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt\right)e^{inx} = \sum_{-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{in(x-t)}dt\right)$$

- Same as

$$\int_{-\pi}^{\pi} f(t)\left(\sum_{-N}^{N} e^{in(x-t)}\right)dt = \int_{-\pi}^{\pi} f(x-t)\left(\sum_{-N}^{N} e^{int}\right)dt$$

period

- $$\boxed{D_N(t) = \sum_{-N}^{N} e^{int}}$$
  is called the Dirichlet Kernel.
- A more useful expression
  $$\boxed{D_N(t) = \frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{t}{2})}}$$

$$\boxed{S_N(x) = \int_{-\pi}^{\pi} f(x-t)\, D_N(t)\, dt}$$

$$\sum_{-N}^{N} e^{inx} = \frac{\sin\left((N+1/2)x\right)}{\sin(x/2)}$$

$$\frac{e^{\frac{ix}{2}}}{e^{-ix/2}}\left(e^{-iNx} + e^{-i(N-1)x} + \cdots + 1 + \cdots e^{i(N-1)x} + e^{iNx}\right)$$

$$-e^{-i(N-1/2)x} + \cdots \qquad e^{i(N+1/2)x}$$

$$e^{-i(N+1/2)x} \cdots \qquad e^{i(N-1/2)x}$$

$$e^{ix/2} D_N(x) - e^{-ix/2} D_N(x) \qquad e^{ix/2} - e^{-ix/2} = 2i \sin x/2$$

$$= e^{i(N+1/2)x} - e^{-i(N+1/2)x} = 2i \sin(N+1/2)x$$

$$\boxed{D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)}}$$

$1/2\pi$

$D_N(T)$ over 3 periods for $N = 3, 5, 7$:

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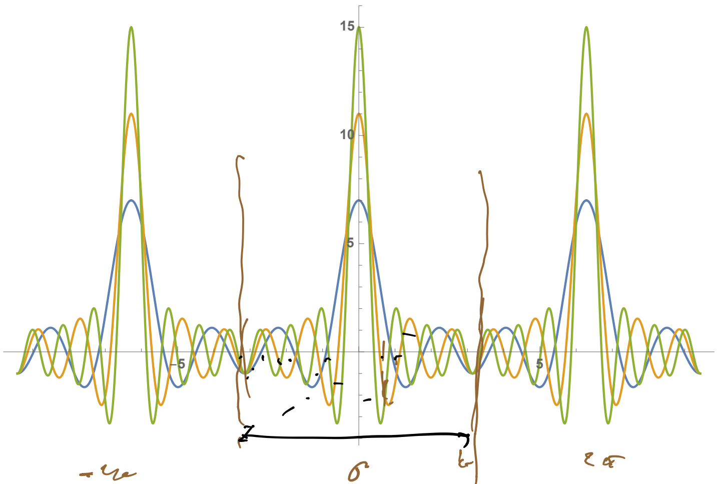


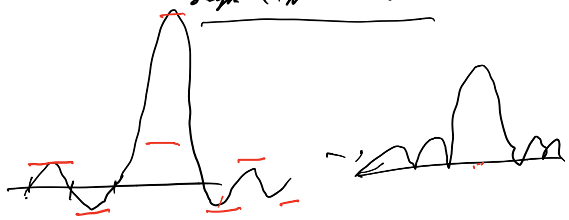
$S_N(f;x) = \frac{1}{2\pi}\int f(x-t)\, D_N(t)$

$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)$ e1

$\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(t)|\, dt$

$\longrightarrow \infty$ as $N \to \infty$

$\mathrm{Sign}\,(D_N(t))\,D_N(t)$

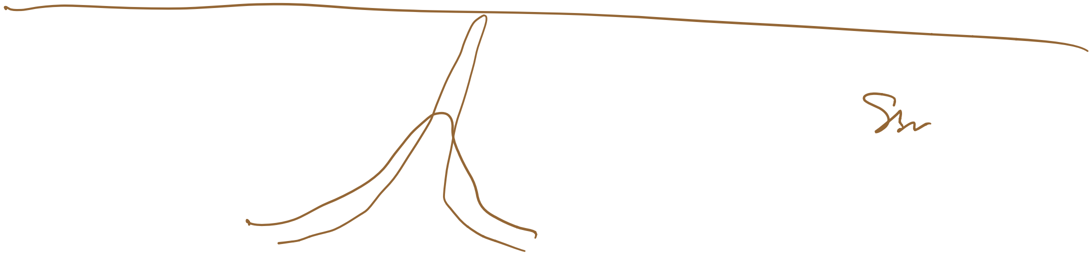


$$\int \mathrm{Sign}\, D_N(x-t)\, D_N(t)\, dt$$

$\|$ $\qquad\qquad \|_\infty$

$\int |D_N(t)| \longrightarrow \infty$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1 \qquad \int \sum_{-N}^{N} e^{inx} = 2\pi$$



S_N

$$S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_N(t)\, dt$$

? $\downarrow$

$f(x)$

$$f(x) = \frac{1}{2\pi}\int f(x)\, D_N(t)\, dt$$

$$f(x) - S_N(f,x) = \frac{1}{2\pi}\int \left(f(x) - f(x-t)\right) D_N(t)\, dt$$

Thm Suppose $x \in [-\pi, \pi]$ s.t.

$\exists\ \delta > 0,\ M > 0$ constants

s.t. $|f(x-t) - f(x)| \le M|t|$

for $|t| < \delta$

$\Rightarrow S_n(f, x) \to f(x)$

$$\left| f(x) - S_N(f,x) \right|$$

$$= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(x) - f(x-t) \right) D_N(t)\, dt \right|$$

$$= \left| \frac{1}{2\pi} \int \left(f(x) - f(x-t) \right) \frac{\sin\left((N+\frac{1}{2}) t \right)}{\sin(t/2)}\, dt \right|$$

$$\left| \frac{1}{2\pi} \int \underbrace{\left(\frac{f(x) - f(x-t)}{\sin(t/2)} \right)}_{} \underbrace{\sin\left(N + \tfrac{1}{2}\right)t}_{} \, dt \right.$$

$\sin Nt \cos t/2 + \cos Nt \sin t/2$

$M(t)$

$$\int \boxed{\frac{f(x) - f(x-t)}{\sin t/2}} \left(\sin Nt \cos t/2 + \cos Nt \sin t/2 \right)$$

M

$\sin t/2 \sim t/2$

M

$$\boxed{\frac{f(x) - f(x-t)}{\sin t/2}} \left(\sin Nt \cos t/2 \right)$$

$f(t)$

bdd

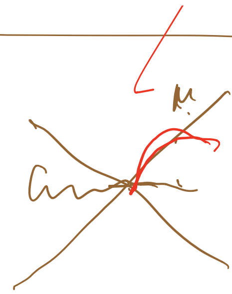
$$\left|\frac{f(x-t)-f(x)}{\sin t/2}\right| \le M$$

$$\int_{-\pi}^{\pi} B(t)\sin Nt + \int b(t)\cos Nt$$

$$\left|\int_{\xi}(\sin Nt)\right| \to 0$$

$$\frac{1}{N} \to 0 \text{ as } N\to\infty$$

see below



"Lipshits"

$$S_N(f,x) = \int f(x-t)\, D_N(t)\, dt$$

Recall:

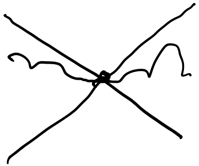
Theorem

Suppose that f is Riemann integrable and that for some x there are constants $\delta > 0$ and $M > 0$ so that

$$|f(x+t) - f(x)| \leq M|t|$$

holds for all $t \in [-\delta, \delta]$. Then

$$\lim_{N \to \infty} s_N(f; x) = f(x).$$



- Recall

$$\boxed{s_N(f;x) = \frac{1}{2\pi}\int f(x-t)D_n(t)dt}$$

where

$$D_N(t) = \frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{t}{2})}$$

is Dirichlet's Kernel, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)dt = 1$$

- Thus

$$\boxed{s_N(f;x) - f(x)} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-t) - f(x))\frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{t}{2})}dt$$

- Write

$$\sin((N+\frac{1}{2})t) = \cos(Nt)\sin(\frac{t}{2}) + \sin(Nt)\cos(\frac{t}{2})$$

- The formula for $s_N(f,x) = f(x)$ is a sum of two terms:
-
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f(x-t)-f(x)}{\sin\frac{t}{2}}\cos(\frac{t}{2})\right)\sin(Nt)dt$$

- and
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-t)-f(x))\cos(Nt)dt$$

$\int (\quad) \sin Nt$ = N^{th} Sine Fourier coeffs of

$$a_0 + \sum a_n \cos nt + \sum b_n \sin nt$$

Bessel's ineq

$$a_0^2 + \underline{\sum a_n^2 + \sum b_n^2} < \infty \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx$$

$\Rightarrow$

$a_n \to 0$

$b_n \to 0$

- The first is the N^{th} Fourier sine coefficient of

$$\frac{f(x-t)-f(x)}{\sin(\frac{t}{2})}\cos(\frac{t}{2})$$

which is Riemann integrable by the assumption $|f(x-t)-f(x)| \leq M|t|$ using $\sin(t) \sim t$
- The second is the N^{th} Fourier cosine coeff of a Riemann integrable function.
- By Bessel's inequality these $\to 0$ as $N \to \infty$.

Fejer's Theorem

- Cesaro sums: given $\{s_n\}$, define

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}$$

- $\{s_n\}$ is Cesaro summable if $\{\sigma_n\}$ converges
- $\{s_n\}$ convergent $\Rightarrow$ Cesaro summable
- Not conversely.

Ex: if $\{S_n\}$ conv → so to σ $\frac{S_0 + \cdots + S_n}{N+1}$

$\forall \varepsilon > 0\ \exists M$ $|S_n - S_\infty| < \varepsilon$ if $n \geq M$

$S_0 + \cdots + S_M + \overline{S_{M+1} \cdots}^{S_\infty \cdot \#}$

$1, -1, 1, -1, 1, -1, \ldots$

$1, 0, 1, 0, 1, \ldots$

$1, 0, 1,$

$S_0 + \ldots + S_N + S_{N+1} + \ldots$

$N+1$

$\sim S_\infty$

$\sim (N-M) \cdot S_\infty + \ldots$

Fejer's **Theorem**

$$\boxed{f \text{ continuous} \Rightarrow \quad \sigma_N(f : x) \to f \text{ uniformly.}}$$

$1, 0, 1, 0, 1, 0$

$1, 1, 2, 2, 3, 3, 4, 4, \ldots \to 1/2$

Reason $S_N(f,x) = \frac{1}{2\pi}\int f(x-t)\, D_N(t)$

$$\sigma_N = \frac{S_0 + S_1 + \cdots + S_N}{N+1} = \frac{1}{2\pi}\int f(x-t)\left(\frac{D_0 + D_1 + \cdots + D_N}{N}\right) dt$$

$$D_N(t) = \frac{\sin (N+\frac{1}{2})t}{\sin t/2}$$

$$\frac{D_0 + \cdots D_{N-1}}{N+1} = \left(\frac{\sin \frac{Nt}{2}}{\sin t/2}\right)^2$$

$D_0 + D_1 + D_2 \cdots$

$$= \frac{\sin(\tfrac{1}{2}t)\,\sin t/2 + \sin(\tfrac{3}{2}t)\,\sin t/2 + \sin(\tfrac{5}{2}t) \cdots}{(\sin t/2)^2}$$

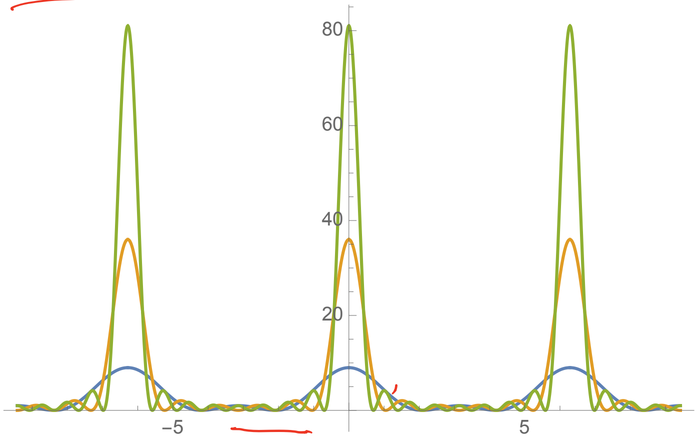
$\sin(n+\frac{1}{2})t$

$\sin nt \cos t/2 + \cos nt \sin t/2$

$$\sum_{0}^{N-1} \sin(t/2)\,\sin(n+\tfrac{1}{2})t = 1 - \cos Nt$$

Fejer's

Out[•]=



just like poly approx

Some $\int_{-\delta}^{\delta} \approx 1$

$|x| > \delta$ small

$$\boxed{\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6}$$

$$1/n^4 = \pi^4(\ )$$

$$1/n \qquad \pi^6$$

---

f periodic on $\mathbb{R}$, Riemann int

$$f \sim \sum c_n e^{inx}$$

$$\boxed{\sum |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx}$$

Gamma Function

- Definition: For $x > 0$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

- Note: for $0 < x < 1$ have to check both 0 and ∞.
- Integration by parts:

$$\boxed{\Gamma(x+1) = x\Gamma(x)}$$

$\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 1$

$\Gamma(2) = \Gamma(1) = 1$

Γ

$\Gamma(3) = 2\Gamma(2) = 2$

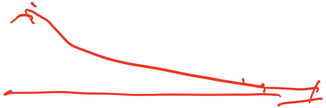
$\Gamma(4) = 3\Gamma(3) = 3!2$

$\vdots$

$\Gamma(n+1) = n!$

extension of $n!$ to $\mathbb{R}^+$

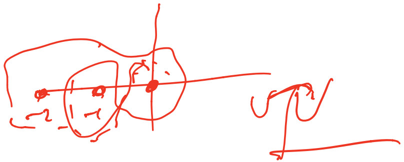
$$\int_0^\infty t^{x-1} e^{-t}\,dt$$



$0 < x < 1$ Converges improper int at 0

$\Gamma(x)$ $\quad$ $\Gamma(z)$ $\quad z \in \mathbb{C}$

$\mathbb{C} - \{\text{negative integers}\}$



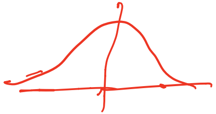
$$\Gamma(1/2) \qquad \int_0^\infty t^{-1/2} e^{-t}\, dt$$

$$\frac{t = s^2}{dt = 2s\,ds} \qquad \int_0^\infty s^{-1} e^{-s^2}\, 2s\, ds$$

$$= 2\int_0^\infty e^{-s^2}\, ds$$

sum

$$= \int_{-\infty}^{\infty} e^{-s^2} ds =$$



$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2$$

$$= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

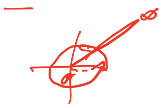
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx\,dy$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

change to polar $\quad x^2+y^2=r^2$

$dx\,dy = \underline{r}\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$



$$= \frac{e^{-r^2}}{-2}\Big|_0^{\infty} = \frac{1}{2}$$

$$0-(-1/2) = 1/2$$

$$\int_0^{2\pi} 1/2\, d\theta = \pi$$

$$\left(\int_{-\infty}^{\infty} e^{-s^2}ds\right)^2 = \pi$$

$$\boxed{\int_{-\infty}^{\infty} e^{-s^2}ds = \sqrt{\pi}}$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^n = (\sqrt{\pi})^n = \pi^{n/2}$$

$$\parallel$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-x_1^2}e^{-x_n^2}\cdots dx_1\cdots dx_n$$

$$= \boxed{\int_{\mathbb{R}^n} e^{-(x_1^2+\cdots+x_n^2)}\,dx_1\cdots dx_n}$$

$R^n : S^{n-1}\times \mathbb{R}^+ \to \mathbb{R}^2$

$v \in S^{n-1}$

$\|v\| = 1$

$r>0$

$v, r \to \underline{r}\boxed{v}$

$$\int_{\mathbb{R}^n} e^{-r^2}$$

(n)

r

"Area" S^{n-1}

"..... vol S^{n-1}

$$= \int_{S^{n-1}}\int_0^{\infty} e^{-r^2}\,\_\_$$

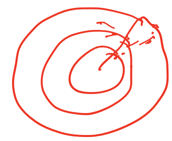
$s_{n-1} = \underline{vol(S^{n-1})} \leftarrow S^{n-1} = \{v\in\mathbb{R}^n ; \|v\|=1\}$

$\underline{b_n} = \underline{vol}\,(B^n) \quad .. \quad B^n = \{v\in\mathbb{R}^n ; \|v\|\le 1\}$

vol in $\mathbb{R}$ $\quad vol(B) = \int_0^1 vol(\|v\|=r)\,dr$

$(S_{n-1})\cdot r^{n-1}$

$\iint$



$$\pi^{n/2} = \boxed{S_{n-1}} \int_0^\infty e^{-r^2} r^{n-1}\,dr$$

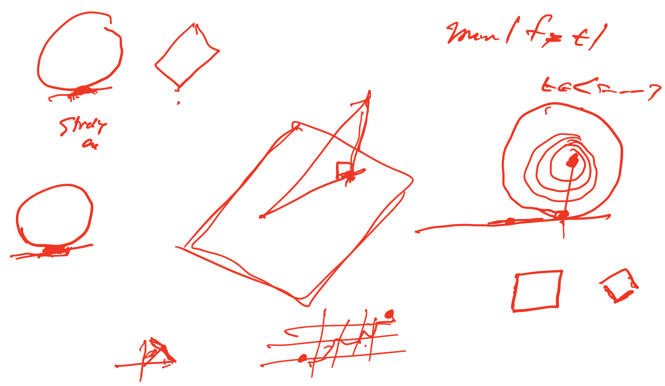
$$\int_0^\infty e^{-r^2} \left(S_{n-1} r^{n-1}\right) dr$$

$$S_{n-1}\left(\quad\right) = \pi^{n/2}$$

Compute S_{n-1} explicitly. $\Gamma \to \pi, \pi!$

b_n

$\max |f_x t|$



Study

$|x| = \sqrt{x\cdot x}$

Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$$

converges $s > 1$

$\zeta(2)$, $\zeta(4)$, ...

$\pi^2/6$

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = \prod (1 - 1/p^s)^{-1}$$

$\Leftrightarrow$ unique factorization

$$\frac{1}{1-1/p^s} = 1 + 1/p^s + 1/p^{2s} + \cdots$$

$$\prod_p \left(1 + 1/p^s + \cdots\right) = \cdots$$

$$=$$

$$1 + 1/p^{rs} = \quad 1/p_1 p_2 \cdots p_r$$