Foundations of Analysis II
Week 8

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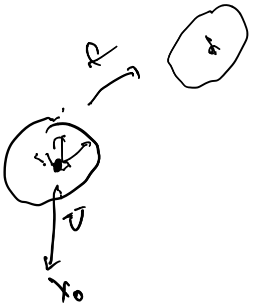
HW 3 is posted

Inverse Function Theorem

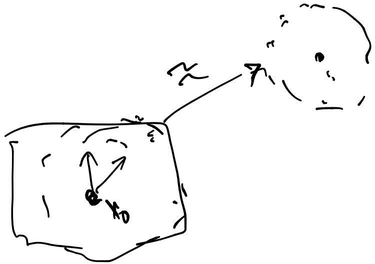
$\mathcal{C}^1$



- $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ continuously differentiable.
- Suppose $x_0 \in U$ and the derivative $d_{x_0} f \in L(\mathbb{R}^n)$ is invertible.
- Then there are neighborhoods N_{x_0} of x_0 and N_{y_0} of $y_0 = f(x_0)$ such that
  - $f(N_{x_0}) = N_{y_0}$ and $f : N_{x_0} \to N_{y_0}$ is bijective.
  - The map $g : N_{y_0} \to N_{x_0}$ inverse to $f|_{N_{x_0}}$ is continuously differentiable

$d_{x_0} f$ invertive

$(d_{x_0} f)^{-1}$ exists



Nbd of p

Means: open set containing p.

Some remarks

- The hypothesis $d_{x_0} f$ invertible is equivalent to the Jacobian matrix

$$\boxed{\left(\frac{\partial f_i}{\partial x_j}(x_0)\right)}$$

  being an invertible n by n matrix.

- From $g(f(x)) = x$ for $x \in N_{x_0}$ and the chain rule if follows that

$$d_{f(x)} g = (d_x f)^{-1} \quad \text{for all } x \in N_{x_0}$$

$$d_{f(x)} f^{-1} = (d_x f)^{-1}$$

- Equivalent statement

$$d_y g = (d_{g(y)} f)^{-1} \quad \text{for all } y \in N_{y_0}$$

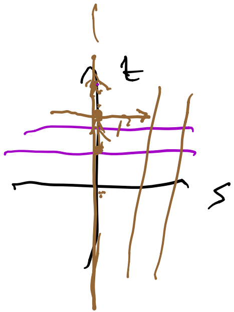
$$(d_y f^{-1}) = (d_{f^{-1}(y)} f)^{-1}$$

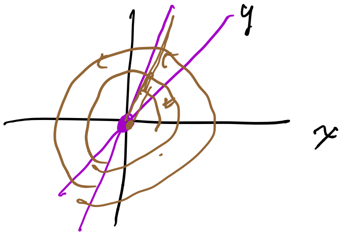
Θ, t

$$d_{(0,t)} f = \begin{pmatrix} \cos t & 0 \\ \sin t & 0 \end{pmatrix}.$$

$d f(e_1)$ $d f(e_2)$

$s > 0$





Example

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(s,t) = (s\cos(t), s\sin(t)) = (x,y) \quad \text{(polar coordinates)}$$

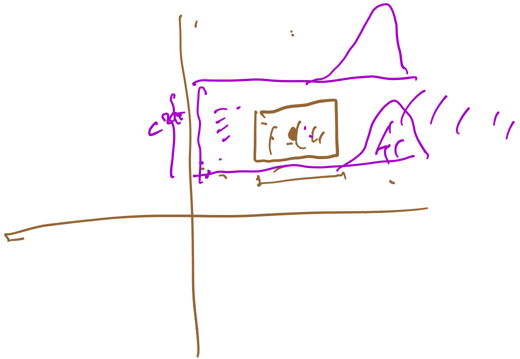
- Jacobian matrix

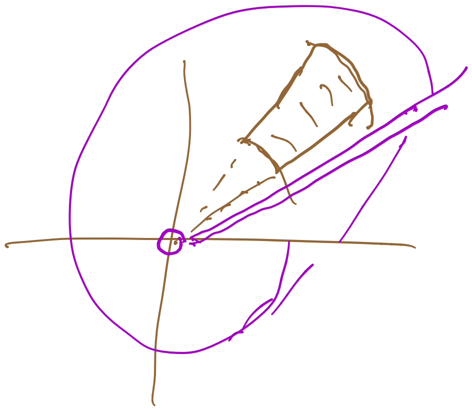
$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{pmatrix}$$

- Invertible if and only if $s \neq 0$ (determinant $= s$)

$$\begin{pmatrix} \cos t & 0 \\ \sin t & 0 \end{pmatrix}$$

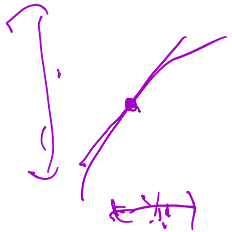
- $f(s, t + 2\pi) = f(s, t)$, so f not globally invertible.
- If (s_0, t_0) has $s_0 > 0$, restriction to $(0, \infty) \times (t_0 - \pi, t_0 + \pi)$ is invertible.





on $s > 0$ map is locally invertible

$s = 0$ not invertible, even locally



Proof of the one variable theorem ($n = 1$)

Jacob matrix = $(f'(x_0))$

- If $f'(x_0) \neq 0$, say $f'(x_0) > 0$, there is an open interval J with $x_0 \in J$ and $f'(x) > \frac{f'(x_0)}{2} > 0$ for all $x \in J$.
- Use

$$\boxed{f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)}$$

for all $x_1 < x_2$ in J and for some $\xi = \xi(x_1, x_2)$ between x_1 and x_2.

- Let $a = \frac{f'(0)}{2}$. Get

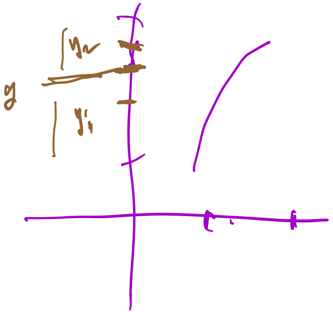
$$f(x_2) - f(x_1) > a(x_2 - x_1) \quad \text{for all } \; x_1 < x_2 \text{ in } J,$$

$y_1 = f(x_1)$

$y_2 = f(x_2)$

$$y_2 - y_1 > a\,(f^{-1}(y_2) - f^{-1}(y_1))$$

$$f^{-1}(y_2) - f^{-1}(y_1) < \frac{1}{a}(y_2 - y_1)$$



$y_1, y_2 \in f(J)$

$y_1 < y < y_2 \Rightarrow y = f(x)$ for some $x \in J$.

Connectedness

$f(J)$ is connected

if $y \notin f(J)$,

Then $f(J)$ disconnected

We get:

- f is injective, so $f^{-1} : f(J) \to J$ exists.
- f^{-1} is continuous:
  Let $y = f(x)$. Then above inequality same as

$$f^{-1}(y_2) - f^{-1}(y_1) < \frac{1}{a}(y_2 - y_1)$$

- $\boxed{f(J) \text{ is an interval: use Intermediate Value Theorem.}}$

- f^{-1} is $\mathcal{C}^1$: Write original equation as

$$y_2 - y_1 = f'(\xi)(f^{-1}(y_2) - f^{-1}(y_1))$$

for some ξ between $f^{-1}(y_1)$ and $f^{-1}(y_2)$

- Let $y_2 \to y_1$. Get

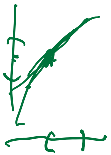
$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

$$a(x_1,x_2) = \begin{cases} \dfrac{f(x_2)-f(x_1)}{x_2-x_1} & x_1 \neq x_2 \\ f'(x_1) & x_1 = x_2 \end{cases}$$

f diff

$$\boxed{f(x_2) - f(x_1) = a(x_1,x_2)(x_2-x_1)}$$

$$\boxed{a(x,x) = f'(x)}$$



$$\boxed{f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)}$$

$$f(x_2)-f(x_1)=\int_{x_1}^{x_2} f'(t)\,dt$$

$$|f(x_2)-f(x_1)| \le \max\left(f'(t) : x_1 \le t \le x_2\right)(x_2-x_1)$$

$$f(x_2)-f(x_1)=$$

$\gamma(t)$

$= (1-t)x_1 + t x_2$

$x_1 \qquad x_2$

$$\frac{d}{dt} f((1-t)x_1+t x_2) = f'(\gamma(t))\gamma'(t)$$

$$= f'(\gamma(t))\, x_2-x_1$$

$$f(x_2)-f(x_1)=\int_0^1 \frac{d}{dt} f(\gamma(t))\,dt$$

$$= \int_0^1 f'(\gamma(t))\,\gamma'(t)\,dt$$

Proof in $n > 1$ variables

$f(x_2) - f(x_1) = \left(\int_0^1 f'(\gamma(t))\,dt\right)(x_2 - x_1)$

- For $n > 1$ it is possible to use the existence of a continuous map $A : U \times U \to L(\mathbb{R}^n)$ such that $f'(\xi)$

$$f(x_2) - f(x_1) = A(x_1, x_2)(x_2 - x_1)$$

to prove the "easier " statements as in the one-variable case.

$\exists\ A : U \times U \longrightarrow L(\mathbb{R}^n) = L(\mathbb{R}^n, \mathbb{R}^n)$

$f(x_2)$ $f(x_1)$

$L(\mathbb{R}^m, \mathbb{R}^n) = \{\text{linear transfs } \mathbb{R}^m \to \mathbb{R}^n\}$

- A possible choice of A is

$$\boxed{A(x_1, x_2) = \int_0^1 d_{\lambda(t)} f \, dt}$$

where $\lambda(t) = \lambda_{x_1,x_2}(t) = (1-t)x_1 + tx_2$ is the straight line segment from x_1 to x_2.

$$\underline{f(x_2) - f(x_1)} = \underline{A(x_1, x_2)}\,\underline{(x_2 - x_1)}$$

.

Whenever $\quad f(x_2) - f(x_1) = \underbrace{a(x_1, x_2)}_{} (x_2 - x_1)$

$(f'(\xi_{x_1, x_2})$

$\int_0^1 f'(\gamma(t))\,dt.$

- Will need $A(x_1, x_2)$ to be defined only for pairs $(x_1, x_2) \in U \times U$ with $|x_2 - x_1|$ small, so only "local convexity" of U is needed. OK for U open.

- Observe that

$$\boxed{A(x, x) = d_x f}$$

$$f(x+h) - f(x) = A(x, x+h)\,h$$

$$= A(x,x)h + \underbrace{(A(x+h,x) - A(x,x))h}$$

$$|\cdot| \qquad | \leq \underbrace{\|A(x+h)-A(x)\|}_{\to 0 \text{ as } h\to 0}\,|h|$$

$$f(x+h) - f(x) = \underbrace{A(x, x+h)}\, h$$

$$= \left[A(x,x) + \left(A(x, x+h) - A(x,x)\right)\right] h$$

$$= A(x,x)h + \underbrace{\left(A(x,x+h) - A(x,x)\right) h}$$

def of norm

$$\le \|A(x,x+h) - A(x,x)\|\,|h|$$

$$= o(|h|)$$

$|Ax| \le \|A\|\,|x|$

because $\|A(x,x+h) - A(x,x)\| \to 0$ as $h \to 0$

A continuous

$$A(x,x) = d_x f$$

$$f(x+h) - f(x) = A(x,x)h + o(|h|)$$

$$\Rightarrow A(x,x) = d_x f$$

If $f(x_2) - f(x_1) = A(x_1,x_2)(x_2 - x_1)$

in fe tiny ----

everything exced in f open set

follows as in 1-varid.

$$\underline{A(x,x) = d_x f}$$

x_0 where $d_{x_0} f$ invertible

$\Rightarrow A(x_0, x_0)$ invertible

$\Rightarrow \exists$ nbd N of x_0 where

$A(x_1,x_2)$ invertible

$x_1, x_2 \in N$

Proof could proceed as follows:

- Let $a = 2||(d_{x_0}f)^{-1}|| = 2||A(x_0, x_0)^{-1}||$.
- Since A is continuous, the set $\Omega \subset L(\mathbb{R}^n)$ is open, and $A(x_0, x_0) = d_{x_0}F \in \Omega$, x_0 has a nbhd N such that $A(x_1, x_2)$ is invertible for all $(x_1, x_2) \in N \times N$.
- Since inversion and norm are continuous, there exists a nbhd N_{x_0} of x_0, contained in N, so that

$$||A(x_1, x_2)^{-1}|| < a \quad \text{for all } x_1, x_2 \in N_{x_0}$$

(*a* as above)

Proof of injectivity

- Let $y_i = f(x_i)$. Then $y_2 - y_1 = A(x_1, x_2)(x_2 - x_1)$
- Apply $A(x_1, x_2)$ to both sides:

$$\boxed{A(x_1, x_2)^{-1}(y_2 - y_1) = x_2 - x_1}$$

- Norms:

$$|x_2 - x_1| \leq \|A(x_1, x_2)^{-1}\||y_2 - y_1| \leq a\,|y_2 - y_1|$$

$y_2 \approx y_1$

- Thus f is injective on N_{x_0}, and its inverse $f^{-1} : f(N_{x_0}) \to N_{x_0}$ is continuous.

Easy: $d_{x_0} f$ inv

$\Rightarrow \exists$ nbd N_{x_0} s.t. $f: N_{x_0} \to \mathbb{R}^n$

injective

and $f^{-1}: f(N_{x_0}) \to N_{x_0}$

is continuous

f injective on N_{x_0}

$\Rightarrow$ bijection onto $\underbrace{f(N_{x_0})}$

hard: $\boxed{\text{open}}$

Image is open

- Proving $f(N_{x_0})$ is open in $\mathbb{R}^n$ is more difficult for $n > 1$.
- Intermediate value theorem rests on:

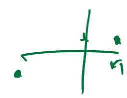
  if J is an open interval in $\mathbb{R}$ and $x \in J$, then $J \setminus \{x\}$ is disconnected.
- If $n \geq 2$, $B \subset \mathbb{R}^n$ is an open ball and $x \in B$, then $B \setminus \{x\}$ is connected.

$$\boxed{f(x_2) - f(x_1) = \underline{A(x_1, x_2)}\,\underline{(x_2 - x_1)}}$$

f class C^1 on U $\iff$ $\exists\ U \times U \xrightarrow{A} L(\mathbb{R}^n)$

$(x_1, x_2) \longrightarrow A(x_1, x_2)$

[illegible]



$A(x,x) = d_x f$

$x_0 \in U$

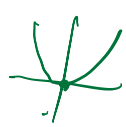
$\exists\ N_{x_0}$ s.t. $f(N_{x_0}) = N_{f(x_0)}$

and $f|_{N_{x_0}}$ is bijective.

prove: $f(N_{x_0})$ is open

$f(x) = x^2$

$-1 \quad 1 \quad f(-1,1) = [0,1)$



- Need more topology.
- Rudin appeals to the *contraction mapping theorem*:
- If (X, d) is a complete metric space, $f : X \to X$ is a *contraction*, that is, there exists a constant $C < 1$ such that

$$d(f(x), f(y)) \leq C\, d(x, y) \quad \text{for all } x, y \in X$$

Then f has a unique fixed point, that is, there is a unique $x_0 \in X$ such that $f(x_0) = x_0$

Complete m

$x \bullet \to f(x) \to f(f(x))$ ⋮

$x_1,\ x_2 = f(x_1)$

$x_3 = f(x_2) = f^2 x_1$

⋮

$\{x_n\}$

Cauchy seq.

$$\underline{d(x_{n+1}, x_n)} < C\, d(x_n, x_{n-1})$$

$$< C^2 d(x_{n-1}, \dots)$$

$$\vdots$$

$$C^{n-1} d(x_2, x_1)$$

$m < n$

$$d(x_m, x_n) \le \underline{d(x_n, x_{n-1})} + \dots + \underline{d(x_{m+1}, x_m)}$$

$\le$ tail end of geom series $\to 0$

Proof of the Contraction Mapping Theorem

- f has at most one fixed point:
  If $f(x_1) = x_1$ and $f(x_2) = x_2$, then

$$d(x_1, x_2)) \leq C\, d(x_1, x_2) \Rightarrow d(x_1, x_2) = 0$$

$d(f(x_1), f(x_2))$

Complete metric

existence

uniqueness: any $C < 1$

- f has a fixed point:
  Pick $x_1 \in X$ and let $x_n = f^{n-1}(x_1)$.
  Since $x_{n+1} = f(x_n)$, $d(x_{n+1}, x_n) < C^{n-1} d(x_2, x_1)$
  if $m < n$, then $d(x_n, x_m) \leq$

  $$d(x_{m+1}, x_m) + \cdots + d(x_n, x_{n-1}) < (C^{m-1} + \cdots + C^{n-2}) d(x_2, x_1)$$

  $\Rightarrow \{x_n\}$ is a Cauchy sequence.
- Let $x_0 = \lim\{x_n\}$. Then

$$f(x_0) = \lim\{x_{n+1}\} = \lim\{x_n\} = x_0$$

$f(x_0) = f(\lim x_n) = \lim(f(x_n)) - \{x_{n+1}\}$

Example of Contraction

$x \to \cancel{x/2}$

$f : \mathbb{R} \to \mathbb{R}$

$|f'| \leq 1/2$

$f : R^n \to \mathbb{R}^n$ of class $\mathcal{C}^1$ and such that

$$||d_x f|| \leq C$$

for all $x \in \mathbb{R}^n$ and for some constant $C < 1$.

$||d_x f|| \leq M$

$|f(x_2) - f(x_1)| \leq ||x_2 - x_1||$

Convert IFT to a FPT

$f: U \to \mathbb{R}^n$

$x_0 \quad (d_{x_0} f)$

- For IFT need to solve an equation

$$\boxed{f(x) = y}$$

$F(x) - y = 0$

$x - x = 0$

- Rewrite

$$x = \boxed{x + (f(x) - y)}$$

- More generally

$$x = x + \boxed{L}(f(x) - y)$$

where L is an invertible linear transformation.

$L(f(x) - y) = 0$

$\Rightarrow f(x) = y$

- For each $y \in \mathbb{R}^n$ and for each invertible $L \in L(\mathbb{R}^n)$, define a map
$$\boxed{\phi = \phi_{y,L} : U \to \mathbb{R}^n}$$
by
$$\phi(x) = x + L(f(x) - y)$$
- Then $f(x) = y \iff \phi(x) = x$
- Challenge: choose $\boxed{L}$ so that we get a contraction of an appropriate complete metric space.

$$\varphi_{y,L}(x) = x + L\left(f(x) - y\right)$$

$$\|d\varphi\| < C < 1$$

$$\boxed{d\varphi = 1 + L\,(df)}$$

$$\|d_x\varphi\| < 1/2$$

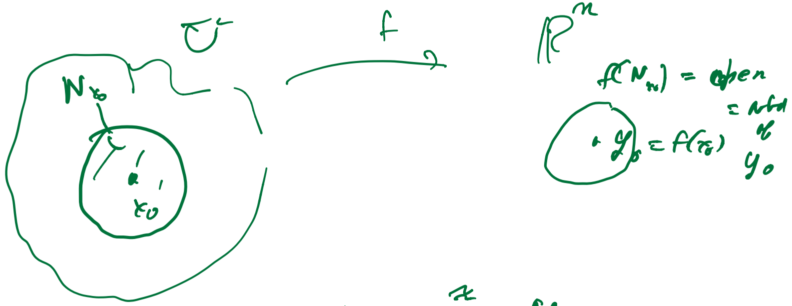
$$\|1 + L\,d_x f\| = L\,(L^{-1}$$

$$L\,(L^{-1} + d_x f$$

Inverse Function Theorem

$\mathscr{C}^1$

- $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ continuously differentiable.
- Suppose $x_0 \in U$ and the derivative $d_{x_0} f \in L(\mathbb{R}^n)$ is invertible.
- Then there are neighborhoods N_{x_0} of x_0 and N_{y_0} of $y_0 = f(x_0)$ such that
  - $f(N_{x_0}) = N_{y_0}$ and $f : N_{x_0} \to N_{y_0}$ is bijective.
  - The map $g : N_{y_0} \to N_{x_0}$ inverse to $f|_{N_{x_0}}$ is continuously differentiable



$$f : N_{x_0} \xrightarrow{\cong} N_{y_0}$$

f^{-1} is C^1

My notation	Rudin
x_0	$\vec{a}$
$d_{x_0} f$	$f'(x_0)$
$A =$	$d_{x_0} f$

$a = \|A^{-1}\|$

$\lambda = \frac{1}{2a}$

Start proof IFT

x_0

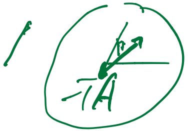
- Let $A = d_{x_0} f$ and let $a = ||A^{-1}||$
- Let

$$N = N_{x_0} = \{x \in U \ : \ ||d_x f - A|| < \frac{1}{2a}\}$$

$$||d_x f - d_{x_0} f|| < \frac{1}{2||(d_{x_0} f)^{-1}||}$$

$1/a$

Recall: A invertible, $||B|| < \frac{1}{a} \Rightarrow A - B$ invertible.



$1/a \longrightarrow$ inv

$1/2a \longrightarrow$ more

- Recall that for any fixed invertible $L \in L(\mathbb{R}^n)$,

$$\boxed{f(x) = y} \iff x = x + L(y - f(x))$$

$L(y - f(x)) = 0$

$\iff y = f(x)$

- In particular

$$f(x) = y \iff x = x + \boxed{A^{-1}}(y - f(x))$$

- For each $y \in \mathbb{R}^n$, define a map $\phi = \phi_y : U \to \mathbb{R}^n$ by

$$\phi_y(x) = x + A^{-1}(y - f(x))$$

- Then $f(x) = y \iff \phi_y(x) = x$

$$\phi = \phi_y$$

- $x \in N \Rightarrow ||d_x \phi|| \leq \frac{1}{2}$

$$\varphi(x) = x + A^{-1}(y - f(x))$$

$$d_x\varphi = \underset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\underset{\shortparallel}{1}} - A^{-1}(d_x f)$$

Step $\frac{1}{2a}$

$$||d_x \varphi|| = ||A^{-1}(A - d_x f)||$$

$$\leq \underbrace{||A^{-1}||}_{a} \, \underbrace{||A - d_x f||}_{< 1/2a}$$

[illegible]

$$\boxed{= 1/2}$$

- $\phi_y : N \to \mathbb{R}^n$ satisfies

$$\boxed{|\phi_y(x_2) - \phi_y(x_1)| \leq \frac{1}{2}|x_2 - x_1|}$$

- $\phi_y : N \to \mathbb{R}^n$ is a contraction (Lipschitz with Lipschitz constant < 1.)

$N \longrightarrow \mathbb{R}^n$

- $\boxed{\cancel{\phi_y} \; f : N \to \mathbb{R}^n \text{ is injective}}$

$f(x_1) = f(x_2) = y$ $\qquad$ $\varphi_y(x_1) = x_1$

$|\varphi(x_1) - \varphi(x_2)| \le \frac{1}{2}|x_1 - x_2|$ $\qquad$ $\varphi_y(x_2) = x_2$

$\varphi \quad \underline{|x_1 - x_2| \le \frac{1}{2}|x_1 - x_2|}$

$\Longrightarrow \quad x_1 = x_2$

$\boxed{\varphi_y(x)}$ $\qquad$ for some $|\varphi_y(x_2) - \varphi_y(x_1)| \le \frac{1}{2}|x_2 - x_1|$

- Note that for fixed x

$$\phi_{y_2}(x) - \phi_{y_1}(x) = A^{-1}(y_2 - y_1)$$

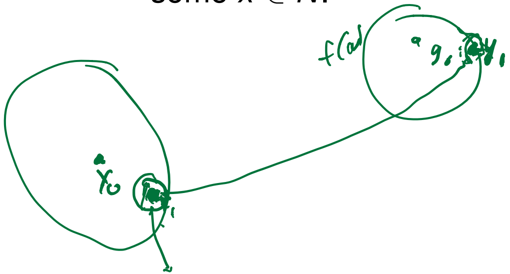
- Thus

$$|\phi_{y_2}(x) - \phi_{y_1}(x)| \leq a|y_2 - y_1|$$

$\phi_y(x)$

$\boxed{\varphi_y(x)}$

- Want to prove $f(N)$ is open in $\mathbb{R}^n$.
- Let $x_1 \in N$ and $y_1 = f(x_1)$.
- Need to find $\rho > 0$ so that $|y - y_1| < \rho \Rightarrow y = f(x)$ for some $x \in N$.



$B(y_1, \rho) \subset f(N)$

- Fix $r > 0$ so that the closed ball $\overline{B(x_1, r)} \subset N$
- Want: $\rho = \frac{r}{2a}$ works.
- First

$\le |a||y_1 - y|$

$$\boxed{|y - y_1| < \frac{r}{2a}} \Rightarrow |\phi_y(x_1) - \phi_{y_1}(x_1)| < \frac{r}{2}.$$

y_1

$f(x_1) = y_1$

$\varphi_{y_1}(x_1) = x_1$

- Next

$$\|x - x_1\| < r \Rightarrow |\phi_y(x) - \phi_y(x_1)| \leq \frac{|x_2 - x_1|}{2}$$

- Together:

$$|y - y_1| < \frac{r}{2a} \text{ and } |x - x_1| \leq r \Rightarrow |\phi_y(x) - x_1| \leq r$$

$r/2c$

y_1

$\forall y \in B(y_1, r/2c)$

$\varphi_y(B(x_1, r)) \subseteq]..$

$|x_1 - x| < r$

$|\varphi_y(x), x_1|$

$|\varphi_y(x) - \varphi_{y_1}(x_1)| + |\varphi_{y_1}(x) - \varphi_{y_1}(x_1)|$ $<$ r

$< r/2$ $< r/2$

- Conclusion:

$$|y - y_1| < \frac{r}{2a} \Rightarrow \phi_y : \overline{B(x_1, r)} \to \overline{B(x_1, r)}$$

- ϕ_y is a contraction of the complete metric space $\overline{B(x_1, r)}$

- Thus there is a unique $x \in \overline{B(x_1, r)}$ with $\phi_y(x) = x$

φ_y all y

only for some specific ...

proved: $f|N_{x_0}$ is injective

$f(N_{x_0})$ is open $= N_{y_0}$

More! $f^{-1}: N_{y_0} \to N_{x_0}$ is C^1

$y_2 - y_1 = \underline{A}(x_1, x_2)(x_2 - x_1)$

N
$A(x_1, x_2)$ invertible $\forall x_1, x_2$ dt.

$A(x_1, x_2)^{-1}(y_2 - y_1) = x_2 - x_1$

f^{-1} is C^1

$C_1, C_2 > 0$

$C_2 |x_2 - x_1| \le |y_1 - y_2| \le C_1 |x_2 - x_1|$

"bi-Lipschitz"

y_1

x_1

1 . 1

$|x_2 - x_1| = \cancel{\text{[illegible]}}\ \underbrace{\|[A(x_1 x_2)]^{-1}\|}_{\ge C > 0} \, |y_2 - y_1|$

$\cancel{|y_2 - x_1|}$

$|x_2 - x_1| \le C \, |y_2 - y_1|$

$|y_2 - y_1| \le C_1 |x_2 - x_1|$

f^{-1} diff ?

Know what $d f^{-1}$ must be

$$d_y f^{-1} = (d_{f^{-1}(y)} f)^{-1}$$

$$f^{-1}(y+k) - f^{-1}(y) = (d_{f^{-1}(y)} f)^{-1}(k) + \ldots$$

$f(N_{x_0})$ open

properties of df at a pt $\longleftrightarrow$ properties of f at nbd of pt

InFT

Inverse Fn Thm

Implicit Fn Thm

$f : U^{\subset \mathbb{R}^n} \longrightarrow \mathbb{R}^n$

diff

if f^{-1} exists & is diff

$\Longrightarrow$ $d_x f$ is invertible

Chain Rule: d

$$f^{-1} \circ f = id$$

$$d \quad f^{-1}(f(x)) = x$$

$$\boxed{\left(d_{f(x)}(f^{-1})\right) \circ d_x f) = 1}$$

$$A, B \in L(\mathbb{R}^n)$$

$$AB = 1$$

$\Rightarrow$ A, B invertible

$A = B^{-1}$

$B = A^{-1}$

$$BA = 1$$

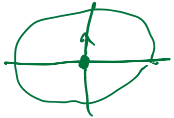
$d_{x_0} f$ invertible

necessary cond for existence of a diff inverse near $f(x_0)$.

§ Simplest case Implicit func thm.

U = nbd of $(0,0)$ in $\mathbb{R}^2$

$f: U \to \mathbb{R}$ C^1, $\begin{cases} \frac{\partial f}{\partial y}(0,0) \neq 0 \\ f(0,0) \end{cases}$



$\Rightarrow \exists$ intervals I, J and a func

$\varphi: I \to J$ s.t. $I \times J \subset U$

$\{(x,y) : f(x,y) = 0\} \cap I \times J$

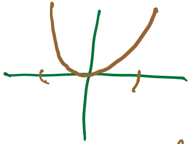
$= \{(x, \varphi(x)) : x \in I\}$

$=$ graph of φ.

$f(x,y)=0$

defines y as implicitly

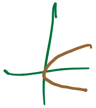
a function of x.



$f(x,y) = y - x^2$

$\frac{\partial f}{\partial y} = 1 \neq 0$

$\varphi(x) = x^2$



$g(x,y) = y^2 - x$

$\frac{\partial g}{\partial y} = 2y = 0$ at $(0,0)$

$y^2 - x$ is <u>not</u> the graph of a fn. in any nbd $(0,0)$

$f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ class C^1

$(\vec{0}, 0)$

$\cancel{\frac{\partial f}{\partial x^{n+1}}(0,0) \neq 0 \Rightarrow \exists\, \varphi : N_1}, N_2$

$\mathbb{R}^n \quad \mathbb{R}$

$d_{(0,0)} f \qquad \mathbb{R}^{n+m} \to \mathbb{R}^n$

$$\left(\cdot^{\,n} \,\middle|\, \overset{m}{\boxed{}} \right) m$$

$\to$ invertible

$\exists\, N_1, N_2, \ \varphi : N_1 \to N_2$ of C^1

$(N_1 \times N_2) \cap (f = 0) = \{ x, \varphi(x) : x \in N_1 \}$

Implicit Function Theorem

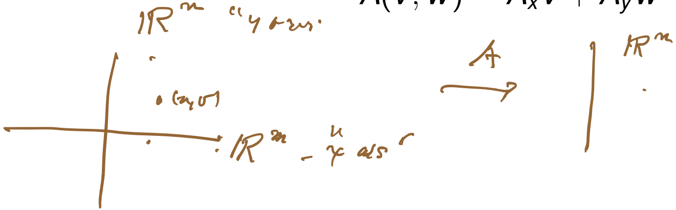
- If $A \in L(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^n)$, write

$$A = (A_x \;\; A_y)$$

Where $A_x \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $A_y \in L(\mathbb{R}^n, \mathbb{R}^n \cancel{Z})$.

- So, if $(v, w) \in \mathbb{R}^m \times \mathbb{R}^n$, $v \in \mathbb{R}^m$, $w \in \mathbb{R}^n$

$$A(v, w) = A_x v + A_y w$$



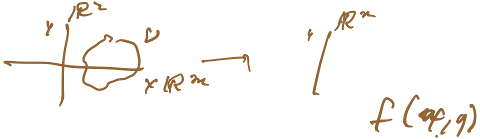
$$A(\underset{\in \mathbb{R}^m}{u}, \underset{\in \mathbb{R}^n}{v}) = \begin{pmatrix} A_x & A_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \underline{A_x}\, u + \underline{A_y}\, v$$

$$A_x : \mathbb{R}^m \to \mathbb{R}^n$$

$$A_y : \mathbb{R}^n \to \mathbb{R}^n$$

$$\overbrace{\begin{pmatrix} \overset{m}{A_x} & \Bigg| & \overset{n}{\underbrace{\boxed{A_y}}_{\text{square}}} \end{pmatrix}}^{m+n} \updownarrow n$$

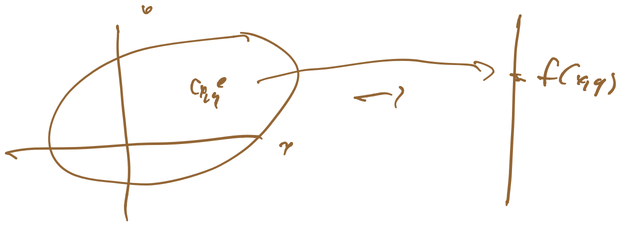


- If $U \subset \mathbb{R}^m \times \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^n$ is differentiable, $(x_0, y_0) \in U$.

$$d_{(x_0,y_0)}f = ((d_{(x_0,y_0)}f)_x \ (d_{(x_0,y_0)}f)_y)) = (\frac{\partial f}{\partial x}(x_0, y_0) \ \frac{\partial f}{\partial y}(x_0, y_0))$$

- Notation not standard
- $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ stand for blocks of the Jacobian matrix of f.

$$\frac{\partial f}{\partial y} = \begin{array}{c} \\ n \end{array} \left(\begin{array}{ccc|ccc} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & & \frac{\partial f_n}{\partial x_m} & \frac{\partial f_n}{\partial y_1} & & \frac{\partial f_n}{\partial y_n} \end{array}\right) n$$



$(x,y) : f(x,y) = 0$

$\left(\frac{\partial f}{\partial y}\right)$ invertible

$\exists\, \varphi : N_1 \to N_2$

s.t. $(x,y) \in N_1 \times N_2$ then $f(x,y) = 0$

$\Leftrightarrow y = \varphi(x)$

Theorem

$\{f(x,y)=0\} \cap (N_1 \times V_1) =$ graph of φ

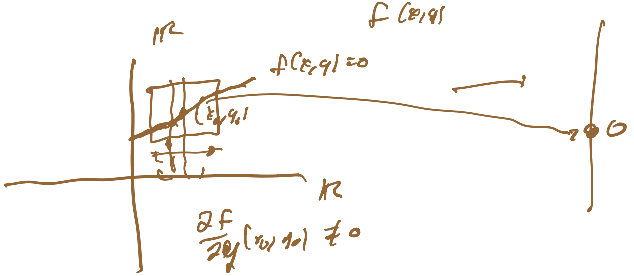
- *$f : U \to \mathbb{R}^n$ as above, f of class $\mathcal{C}^1$.*
- *$(x_0, y_0) \in U$, with $x_0 \in \mathbb{R}^m$ and $y_0 \in \mathbb{R}^n$*
- *Suppose that*
  - *$f(x_0, y_0) = 0$*
  - *$\frac{\partial f}{\partial y}(x_0, y_0) \in L(\mathbb{R}^n)$ is invertible*
- *Then there exist*
  - *Nbs N_x, N_y of x_0, y_0 respectively, with $N_x \times N_y \subset U$,*
  - *A map $\phi : N_x \to N_y$ of class $\mathcal{C}^1$,*
- *Such that*

$$\{(x,y) \in N_x \times N_y \ : \ f(x,y) = 0\} = \{(x, \phi(x)) \ : \ x \in N_x\}$$

Picture for $m = n = 1$

$f(x, y) = 0$ defines y implicitly as a function of x,

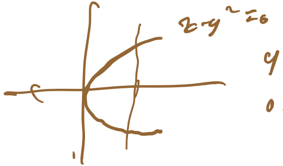
namely $y = \phi(x)$



ex $f(x,y) = y^2 - x$

$(0,0)$

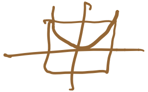
$\frac{\partial f}{\partial y}(0,0) = 2y\big|_{(0,0)} = 0$



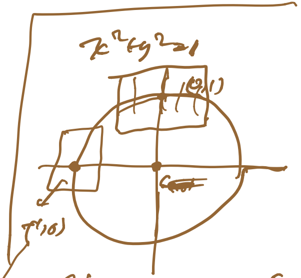
y not a func of x
on any nbd of 0

$f(x,y) = y - x^2$

$y - x^2 = 0 \qquad y = x^2$



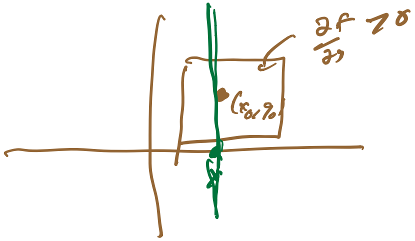
$\frac{\partial f}{\partial y} = 1$



$\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial y} = 2y = 2$ at $(0,1)$

Proof for $m = n = 1$



$f(x_0, y_0) = 0$

$\frac{\partial f}{\partial y}(x_0, y_0) \neq 0 \Rightarrow \begin{cases} > 0 \\ < 0 \end{cases}$

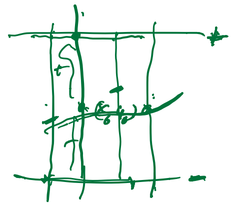
Suppose $\frac{\partial f}{\partial y}(x_0, y_0) > 0$

$\Rightarrow \exists\ N_1 \times N_2$ of (x_0, y_0)

$\frac{\partial f}{\partial y}(x, y) > 0$

$\forall (x, y) \in N_1 \times N_2$

for fixed $x \in N_1$ $f(x, y)$ strictly increasing func of y



for each x $\exists$ y s.t. $f(x,y)=0$

y is unique because

$f(x,y) \nearrow y$

$x \to \varphi(x) =$ unique y in $N_1 \times N_2$

s.t. $f(x,y)=0$

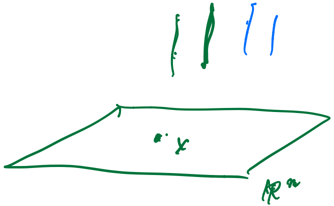
Show φ is C^1

$\frac{d}{dt}\left(\left(f(t,\varphi(t)\right)\right) = 0)$

$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\varphi'(t) = 0$

$$\boxed{\varphi'(t) = \frac{-\frac{\partial f}{\partial x}(t,\varphi(t))}{\frac{\partial f}{\partial y}(t,\varphi(t))}}$$

Same for m arbitrary $n = 1$



$\mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$

$f(x, y)$

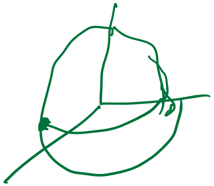
$\uparrow$ 1-dim

for each $(x, y) \in N_1 \times N_2$ $\quad f(x, y) \uparrow y$

Examples

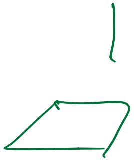
$x^2+y^2+z^2-1=0$

$\frac{\partial f}{\partial z} = 2z \neq 0$ $\quad z=0$



$(0,0,1)$

$z=\sqrt{1-x^2-y^2}$ func of x,y



Inverse Func Thm $\Rightarrow$ Implicit Func Thm

- Define $F : U \to \mathbb{R}^m \times \mathbb{R}^n$ by

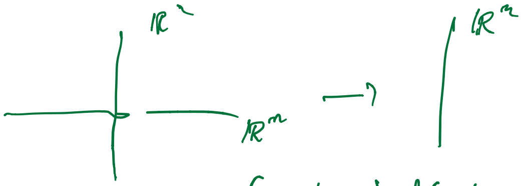
$$\boxed{F(x, y) = (x, f(x, y))}$$

- Then

$$\boxed{d_{(x_0,y_0)}F = \begin{pmatrix} \mathbf{I} & 0 \\ \frac{\partial f}{\partial x} & \boxed{\frac{\partial f}{\partial y}} \end{pmatrix}}$$ invertible

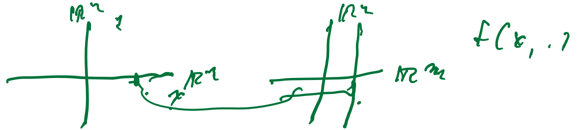
$\Longrightarrow$ $\frac{\partial f}{\partial y}$ invertible

is invertible.

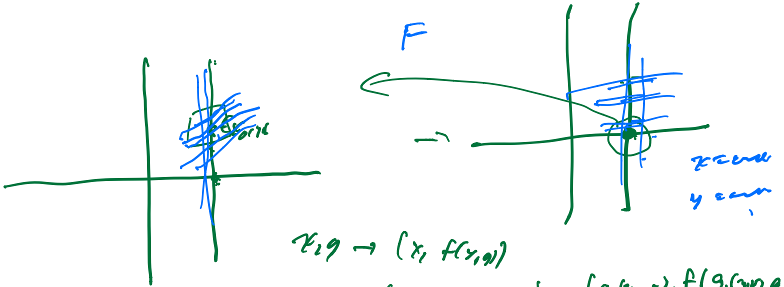


$(x, y) \mapsto f(x,y)$

$F\ (x,y) \to (x, f(x,y))$



- Inverse Function Thm gives local inverse G defined near $F(x_0, y_0) = (x_0, 0)$
- Check $G(u, v) = (u, g(u, v))$ with $f(u, g(u, v)) = v$
- Let $\phi(x) = g(x, 0)$.



$x, y \to (x, f(x,y))$

$x = \text{const}$

$f(x,y) = \text{const}$

$G(u,v) = (g_1(u,v), g_2(u,v)) \to (g_1(u,v), f(g_1(u,v), g_2(u,v)))$
$= (u, v)$

$g_1(u,v) = u$

$G(u,v) = (u, g(u,v))$

$f(u, g(u,v)) = (u,v)$

$\varphi(x) = g(x,0)$

$f(x, g(x,0)) = 0$

Implicit Func Thm $\Rightarrow$ Inverse Func Thm

- Let $f : U \to \mathbb{R}^n$ and $y_0 \in U$ as in Inverse function thm.
- Define $F : \mathbb{R}^n \times \mathbb{R}^n$ by

$$\boxed{F(x, y) = f(y) - x} = 0$$

- Then $F(f(y_0), y_0) = 0$ and $\frac{\partial F}{\partial y}(f(y_0), y_0) = d_{y_0} f$ invertible.
- Then $F(x, \phi(x)) = f(\phi(x)) - x = 0 \Leftrightarrow f(\underline{\phi(x)}) = x$

$y^2 - x$ $\quad$ $x = \sqrt{y}$

$\mathbb{R}^n \to \mathbb{R}^n$ $\quad$ $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$

Lin... $\quad$ Implicit

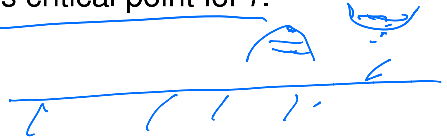
Critical Points

$d_p f : \mathbb{R}^n \to \mathbb{R}$ $\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

$(d_p f)(h)$

$\nabla_p f \in \mathbb{R}^n$ $\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

- $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}$ differentiable.
- $p \in U$ is called a *critical point* of f if $\boxed{d_p f} = 0$ $\quad (d_p f)(h) = (\nabla_p f) \cdot h$
- Equivalently: $\boxed{\frac{\partial f}{\partial x_1}(p) \cdots = \frac{\partial f}{\partial x_n}(p) = 0}$
- Equivalently: Gradient $\boxed{\nabla_p f} = 0.$ ← vector in $\mathbb{R}^n$
- Know (homework): p a local maximum (or min) for $f \Rightarrow p$ is critical point for f.



max or min prob for

$f : U \xrightarrow{} \mathbb{R}$ ($U \subset$ open)

look for critical pts $\nabla_p f = 0$

$(\Leftrightarrow d_p f = 0)$

often

Non-singular hypersurfaces

dim $n-1$ in $\mathbb{R}^n$

(codimension 1)

- Suppose $g : U \to \mathbb{R}$ of class $\mathcal{C}^1$.
- The set $\{g = 0\}$ is called a non-singular *hypersurface* in U. ??
- Suppose that $d_p g \neq 0$ for all $p \in \{g = 0\}$.
- This means that for each $p \in \{g = 0\}$, for at least one $i \in \{1, \ldots, n\}$, $\frac{\partial g}{\partial x_i} \neq 0$.
- By the implicit function thm, each $p \in \{g = 0\}$ has a neighborhod N_p for which one x_i is a $\mathcal{C}^1$-function of the remaining ones.

- To avoid complicated notation, suppose $\frac{\partial g}{\partial x_n}(p) \neq 0$.
- The p has a nbd $N = N_1 \times N_2$, $N_1 \subset \mathbb{R}^{n-1}$, $N_2 \subset \mathbb{R}$. and a $\mathcal{C}^1$ function $\phi : N_1 \to N_2$ such that

$$\{g = 0\} \cap (N_1 \times N_2)$$

is the graph of ϕ

$$\{(x_1, \ldots, x_{n-1}, \phi(x_1, \ldots, x_{n-1})) \ : (x_1, \ldots, x_{n-1}) \in N_1\}$$

- Conclusion: $\{g = 0\} \cap (N_1 \times N_2)$ is in bijective, $\mathcal{C}^1$ correspondence with the open set $N_1 \subset \mathbb{R}^{n-1}$
- Locally $\{g = 0\}$ is an open set in $\mathbb{R}^{n-1}$.
- Called *non-singular hypersuface* for this reason.
- Locally looks like $\mathbb{R}^{n-1} \subset \mathbb{R}^n$

Examples

$x^2+y^2+z^2-1 = g(x,y,z)$

$g=0$: unit sphere

$dg = (2x, 2y, 2z)$

$dg = (2x \quad 2y \quad 2z)$

$\nabla g = (2x, 2y, 2z)$

$\nabla g = 0 \iff x = y = z = 0$

$\Rightarrow x^2+y^2+z^2-1 = -1 \neq 0$

$\nabla g \neq 0$ on $g(x,y,z)=0$.

at least one of $\partial g/\partial x, \partial g/\partial y, \partial g/\partial z \neq 0$

by implicit function

$x = \text{fun}(y,z)$

or $y = f(x,z)$

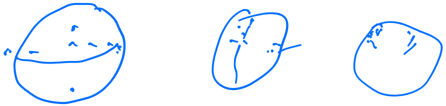
$z = f(x,y)$

$x^2+y^2+z^2-1=0$

$z = \pm\sqrt{1-x^2-y^2}$

or $x = \pm\sqrt{1-y^2-z^2}$

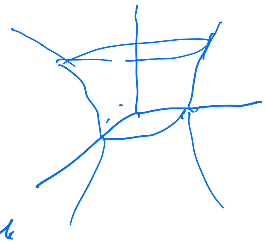
$y = \pm\sqrt{1-x^2-z^2}$



$x^2+y^2-z^2=1$

$x^2+y^2=1+z^2$

hyp of one sheet



non-sing hypersurface

$\frac{\partial g}{\partial x}=2x$ $\quad \frac{\partial g}{\partial y}=2y$ $\quad \frac{\partial g}{\partial z}=-2z$

not all zero $g(x,y,z)=0$

$x^2+y^2-z^2=0$

$z=\pm\sqrt{x^2+y^2}$

Singular surface $\quad$ a singular pt

∇

$U\subset\mathbb{R}^n$

$g: U\to\mathbb{R}$ $\quad \nabla_p g\neq 0$ on $\{g(p)=0\}$

non sing hypersurface

$f: U\to\mathbb{R}$ $\quad C^1$-fn.

critical points of $f\big|_{\{g=0\}}$?

$f(x,y,z)=z$

on $g(x,y,z)=x^2+y^2+z^2-1=0$

any concept involving differentiability can be studied by restrict to open sets in $\mathbb{R}^{n+1}$ domain of the funct single fn. th.

cover your non-sing hypersurface by sets

$\mathbb{R}^{n-1} \times \mathbb{R}$
$\cup$
$V \to \mathbb{R}$

$f|_{\{g=0\}}$ critical point of $f|_{(x,\varphi(x))}$

$= f(x,\varphi(x))$ $x \in N$ in $\mathbb{R}^{n-1}$

crit pts of

$f(x,y,\sqrt{1-x^2-y^2})$, $f(x,y,-\sqrt{1-x^2-y^2})$

$f(\sqrt{1-y^2-z^2},y,z)$, $(-$ $)$

---

z crit pts of z?

$(x,y,\pm\sqrt{1-x^2-y^2})$

$(x,\pm\sqrt{1-x^2-z^2},z)$

$(\pm\sqrt{1-y^2-z^2},y,z)$

z : $\sqrt{1-x^2-y^2}$

$-\sqrt{1-x^2-y^2}$

$\frac{\partial z}{\partial x} \neq 0$

z 1

z

crit pts $(0,0,1)$

$(0,0,-1)$

---

how to do this efficiently

$d\, f \circ (x,\varphi(x)) = 0$

$(d_p f)(\underbrace{\text{vectors tangent to } S \text{ at } p})$

$d(x,\varphi(x))$ $\{ e_i \mapsto \frac{\partial \varphi}{\partial x_i}$

$(0,1,\ldots$ $)$!

$(\nabla_p g)^{\perp}$

$d_p f \,|\, (\nabla_p g)^{\perp} \equiv 0$

$\nabla_p f \perp ((\nabla_p g)^{\perp})$

Critical points of $f|_{\{g=0\}}$

$$\nabla_p f \parallel \nabla_p g$$

$$\Leftrightarrow \exists \lambda \text{ s.t.}$$

$$\boxed{\nabla_p f = \lambda \underbrace{\nabla_p g}_{\neq 0}}$$

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z on $g(x,y,z) = \ldots - 1 = 0$

$$(0,0,1) = \lambda(2x, 2y, 2z)$$

$$x = y = 0 \qquad x^2 + y^2 + z^2 = 1$$

$z^2 = 1$

$z = \pm 1$

$(0, 0, \pm 1)$