MATH 3220-3 HOMEWORK 1

DUE JANUARY 23

(1) Let $f_n(x) = x^n \in \mathcal{C}([0,1])$
  (a) Prove that the sequence $\{f_n\}$ has no convergernt subsequence (in the norm of $\mathcal{C}([0,1])$).
  (b) Use this to prove that the unit ball $\{f \in \mathcal{C}([0,1]) : ||f|| \leq 1\}$ is not compact.

(2) (a) Let (X, d) be a metric space and let $K \subset X$ be a compact subset. Prove that for all $\epsilon > 0$ there are finitely many points $x_1, \ldots, x_n \in K$ so that, for every $x \in K$ there exists an $i, i = 1, \ldots, n$, such that $d(x, x_i) < \epsilon$
  (b) Use this to prove that, if $K \subset \mathcal{C}([0,1])$ is compact, then K is equicontinuous.

(3) Recall the norms $||f||_1 = \int_0^1 |f(x)|\, dx$ and $||f||_\infty = \sup_{x\in X}\{|f(x|)\}$ on $\mathcal{C}([0,1])$.
  (a) Prove that $||f||_1 \leq |||f||_\infty$ for all $f \in \mathcal{C}([0,1])$
  (b) Prove that there is no constant $C > 0$ such that $||f||_\infty \leq C||f||_1$ holds for all $f \in \mathcal{C}([0,1])$ by producing a sequence $f_n \in \mathcal{C}([0,1])$ with $||f_n||_\infty \to \infty$ and $||f_n||_1 = 1$
  (c) Prove that $\mathcal{C}([0,1])$ with norm $||f||_1$ is not a complete metric space. Observe that this gives another proof of (b).
  Suggestion Consider (in $\mathcal{C}([-1,1])$ for simpler formulas) the sequence

$$f_n(x) = \begin{cases} -1 & \text{if } -1 \leq x \leq 1/n, \\ nx & \text{if} -1/n \leq x \leq 1/n, \\ 1 & \text{if } 1/n \leq x \leq 1. \end{cases}$$

(4) This problem and the next are part of Rudin, Chapter 7, Exercise 4. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

  (a) For which $x \in \mathbb{R}$ does the series converge?
  Note: if for a given $x \in \mathbb{R}$ some term in the series is not defined, then the series does not converge for that x.
  (b) For which $x \in \mathbb{R}$ does it converge absolutely?
  (c) Is f bounded?

(5) Using the same series as in the last problem
  (a) For which intervals in $\mathbb{R}$ does the series converge uniformly?
  (b) For which intervals in $\mathbb{R}$ does the series converge, but not uniformly?.