Math 3010 § 1. **Final Exam** Name: Practice Problems
Treibergs March 30, 2018

Here are some problems soluble by methods encountered in the course. I have tried to select problems ranging over the topics we've encountered. Admittedly, they were chosen because they're fascinating to me. As such, they may have solutions that are longer than the questions you might expect on an exam. But some of them are samples of homework problems. Here are a few of my references.

References.

- Carl Boyer, *A History of the Calculus and its Conceptual Development,* Dover Publ. Inc., New York, 1959; orig. publ. *The Concepts of the Calculus, A Critical and Historical Discussion of the Derivative and the Integral,* Hafner Publ. Co. Inc., 1949.
- Carl Boyer, *A History of Mathematics,* Princeton University Press, Princeton 1985
- Lucas Bunt, Phillip Jones, Jack Bedient, *The Historical Roots of Elementary mathematics,* Dover, Mineola 1988; orig. publ. Prentice- Hall, Englewood Cliffs 1976
- David Burton, *The History of Mathmatics, An Introduction*, 7th ed., McGraw Hill, New York 2011
- Ronald Calinger, *Classics of Mathematics,* Prentice-Hall, Englewood Cliffs 1995; orig. publ. Moore Publ. Co. Inc., 1982
- Victor Katz, *A History of Mathematics An introduction,* 3rd. ed., Addison-Wesley, Boston 2009
- Morris Kline, *Mathematical Thought from Ancient to Modern Times,* Oxford University Press, New York 1972
- John Stillwell, *Mathematics and its History, 3rd ed.,* Springer, New York, 2010.
- Dirk Struik, *A Concise History of Mathematics,* Dover, New York 1967
- Harry N. Wright, *First Course in Theory of Numbers,* Dover, New York, 1971; orig. publ. John Wiley & Sons, Inc., New York, 1939.

---

1. *Use Fermat's method of* Adequality *to find $x > 0$ where $f(x)$ has a local maximum, where*

$$f(x) = \frac{\sqrt{x}}{1+x^2}.$$

Fermat assumes that x and $x+E$ bracket the local max and have equal function values. Thus

$$\frac{\sqrt{x}}{1+x^2} = \frac{\sqrt{x+E}}{1+(x+E)^2}$$

which implies by squaring

$$\begin{aligned}
\left(1+(x+E)^2\right)^2 x &= \left(1+x^2\right)^2 (x+E)\\
\left(1+x^2+(2x+E)E\right)^2 x &= \left(1+x^2\right)^2 (x+E)\\
\left(1+x^2\right)^2 x + 2\left(1+x^2\right)(2x+E)Ex + (2x+E)^2E^2 &= \left(1+x^2\right)^2 x + \left(1+x^2\right)^2 E\\
2\left(1+x^2\right)(2x+E)E + (2x+E)^2E^2x &= \left(1+x^2\right)^2 E
\end{aligned}$$

Dividing by E yields

$$2\left(1+x^2\right)Ex+(2x+E)^2Ex=(1+x^2)(1-3x^2)$$

Neglecting the E terms which vanish (in the limit),

$$0=1-3x^2 \tag{1}$$

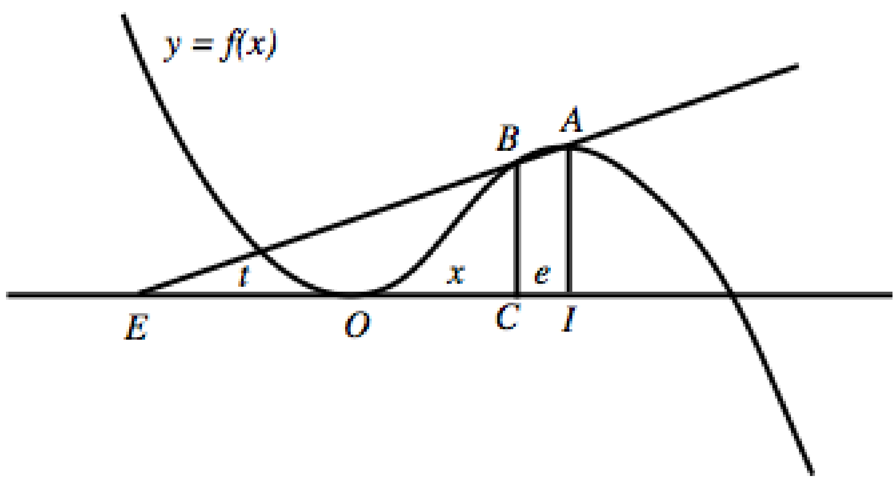
the roots are $x=\pm\frac{1}{\sqrt{3}}$ and the max occurs when $x=\frac{1}{\sqrt{3}}$. As a reality check, we find the max using calculus. The derivative

$$\begin{aligned} f'(x) &= \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+x^2}-\frac{2x\sqrt{x}}{\left(1+x^2\right)^2} \\ &= \frac{\frac{1}{2}\left(x^{-\frac{1}{2}}+x^{\frac{3}{2}}\right)-2x^{\frac{3}{2}}}{\left(1+x^2\right)^2} \\ &= \frac{1-3x^2}{2\sqrt{x}\left(1+x^2\right)^2} \end{aligned}$$

Thus again, $f'(x)=0$ when (1) holds so the maximizing point is the one computed already.

2. *Use Fermat's method of* Ad-equality *to find the tangent line and slope of the function at* x.

$$f(x)=x^2-x^3.$$



We seek the tangent line at $B=(x,f(x))$. A nearby point on the curve is $A=(x+e,f(x+e))$ where $e=CI$. Let $t=EO$. This time A and B bracket the point of tangency. Thus Fermat ad-equated the slopes of the triangles $\triangle BCE$ and $\triangle AIE$

$$\frac{BC}{CE}=\frac{AI}{IE}$$

yielding

$$\frac{f(x)}{t+x}=\frac{f(x+e)}{t+x+e}.$$

Substituting the function,

$$\frac{x^2-x^3}{t+x}=\frac{(x+e)^2-(x+e)^3}{t+x+e}=\frac{x^2-x^3+(2x-3x^2)e+(1-3x)e^2-e^3}{t+x+e}.$$

Cross multiplying yields

$$\left(x^2 - x^3\right)(t + x + e) = \left(x^2 - x^3 + (2x - 3x^2)e + (1 - 3x)e^2 - e^3\right)(t + x).$$

Cancelling,

$$\left(x^2 - x^3\right)e = \left((2x - 3x^2)e + (1 - 3x)e^2 - e^3\right)(t + x).$$

Dividing by e and collecting terms yields

$$x^2 - x^3 - (2x - 3x^2)(t + x) = \left((1 - 3x)e - e^2\right)(t + x).$$

Neglecting e terms gives

$$x^2 - x^3 - (2x - 3x^2)(t + x) = 0.$$

In other words, the slope of the tangent line is

$$\frac{f(x)}{t + x} = \frac{x^2 - x^3}{t + x} = 2x - 3x^2$$

and

$$t = \frac{x^2 - x^3 - (2x - 3x^2)x}{2x - 3x^2} = \frac{2x^2 - x}{2 - 3x}.$$

Of course, this agrees with the modern computations

$$f'(x) = 2x - 3x^2$$

and since the slope is the derivative,

$$t = \frac{f(x)}{f'(x)} - x = \frac{x^2 - x^3}{2x - 3x^2} - x = \frac{2x^2 - x}{2 - 3x}.$$

3. *Use Fermat's method of Ad-equality to find the slope of $f(x) = x^2 - \sqrt{x}$ at $x > 0$. [University of Utah, Math 3010 homework problem, April 13, 2018.]*

The tangent line is approximated by a secant line through two infinitesimally close points at $A = (x, f(x))$ and $B = (x + E, f(x + E))$. Let $T = (t, 0)$ be the intersection point of the tangent line with the x-axis and $C = (x, 0)$ and $D = (x + E, 0)$. Then the slopes of the two similar triangles $\triangle ACT \sim \triangle BDT$ are equal

$$\frac{f(x)}{x - t} = \frac{f(x + E)}{x + E - t}.$$

Cross multiplying

$$(x^2 - \sqrt{x})(x + E - t) = f(x)(x + E - t) = f(x + E)(x - t) = ((x + E)^2 - \sqrt{x + E})(x - t).$$

Isolate the square root

$$(x^2 - \sqrt{x})(x + E - t) - (x + E)^2(x - t) = -(x - t)\sqrt{x + E}.$$

Square

$$\left[(x^2 - \sqrt{x})(x + E - t) - (x + E)^2(x - t)\right]^2 = (x - t)^2(x + E)$$

$$(x^2 - \sqrt{x})^2(x + E - t)^2 - 2(x^2 - \sqrt{x})(x - t)(x + E - t)(x + E)^2 + (x + E)^4(x - t)^2$$
$$= (x - t)^2(x + E)$$

Expand in E

$$\begin{gathered}(x^2-\sqrt{x})^2\big[(x-t)^2+2(x-t)E+E^2\big]\\ -2(x^2-\sqrt{x})(x-t)(x+E-t)\big[x^2+2xE+E^2\big]\\ +\big[x^4+4x^3E+6x^2E^2+4xE^3+E^4\big](x-t)^2=(x-t)^2(x+E).\end{gathered}$$

Write A, B, C, D, G, H for functions of x and t,

$$\begin{gathered}(x^2-\sqrt{x})^2(x-t)^2+2(x^2-\sqrt{x})^2(x-t)E+AE^2\\ -2(x^2-\sqrt{x})(x-t)^2x^2-2(x^2-\sqrt{x})(x-t)\big[x^2+2x(x-t)\big]E+BE^2+CE^3\\ +x^4(x-t)^2+4x^3(x-t)^2E+DE^2+GE^3+HE^4=(x-t)^2x+(x-t)^2E\end{gathered}$$

We note that the E^0 terms cancel

$$\begin{aligned}(x^2-\sqrt{x})^2(x-t)^2-2(x^2-\sqrt{x})(x-t)^2x^2+x^4(x-t)^2\\ =(x-t)^2\big[(x^2-\sqrt{x})-x^2\big]^2=(x-t)^2x.\end{aligned}$$

We divide out E and drop any remaining terms with a factor E.

$$2(x^2-\sqrt{x})^2(x-t)-2(x^2-\sqrt{x})(x-t)\big[x^2+2x(x-t)\big]+4x^3(x-t)^2=(x-t)^2.$$

If $x\neq t$ which happens except when $x=1$ and $f(1)=0$ we may divide by $x-t$

$$2(x^2-\sqrt{x})^2-2(x^2-\sqrt{x})x^2=\big[1+4x(x^2-\sqrt{x})-4x^3\big](x-t).$$

which implies

$$x-t=-\frac{2(x^2-\sqrt{x})\sqrt{x}}{1-4x\sqrt{x}}.$$

Hence the slope is

$$\frac{f(x)}{x-t}=-\frac{1-4x\sqrt{x}}{2\sqrt{x}}=2x-\frac{1}{2\sqrt{x}}.$$

Of course this agrees with $f'(x)$ computed the modern way. This function is contnuous at $x>0$ so it continues to provide the derivative at $x=1$. $x-t$ blows up when $1-4x\sqrt{x}=0$ at $x=2^{-4/3}$. Since $f(2^{-4/3})\neq 0$ this simply means that the slope is zero at this point. Indeed, it is a minimum point for $f(x)$.

Bo Zhu recommends the following procedure. Instead of isolating the square, rewrite the equation as

$$\begin{gathered}x^2(x+E-t)-(x+E)^2(x-t)+(\sqrt{x+E}-\sqrt{x})(x-t)-\sqrt{x}E=0\\ x^2(x-t)+x^2E-(x^2+2xE+E^2)(x-t)+\\ +(\sqrt{x+E}-\sqrt{x})\frac{\sqrt{x+E}+\sqrt{x}}{\sqrt{x+E}+\sqrt{x}}(x-t)-\sqrt{x}E=0\\ x^2E-(2xE+E^2)(x-t)+\frac{E}{\sqrt{x+E}+\sqrt{x}}(x-t)-\sqrt{x}E=0\end{gathered}$$

Divide by E and set the remaining $E=0$.

$$x^2-2x(x-t)+\frac{1}{2\sqrt{x}}(x-t)-\sqrt{x}=0$$

which implies

$$\frac{x^2-\sqrt{x}}{x-t}=2x-\frac{1}{2\sqrt{x}}$$

as before.

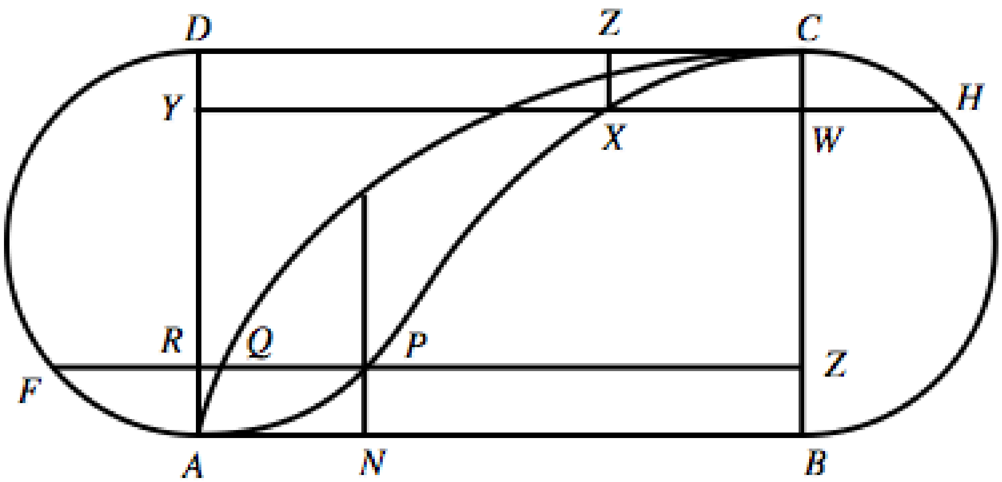
4. *Following Roberval, find the are under one half of one arch of the cycloid*

$$x(t) = a(t - \sin t); \qquad y(t) = a(1 - \cos t)$$

generated by following a point on the edge of a circle of radius a as it rolls along the x-axis. First show using Cavalieri's principle that the area between the cycloid and its companion curve

$$x(t) = at, \qquad y(t) = a\,(1 - \cos t)$$

equals half the area of its generating circle. Then show, again using Cavalieri's principle that the companion curve bisects the area of the rectangle containing the half cycloid arch.



The cycloid AQC is generated by rolling the circle of radius a to the right. As the circle rolls through arc AF, the base of the wheel moves a distance AN. The point that was at A and rolled to Q has moved up to height F above line AB. The distance Q is behind the contact point N is PQ which equals the distance FR. Thus the horizontal slice QP of the area between the cycloid and the companion curve $APXC$ has the same length as the horizontal slice RF of the half disk $ARYDFA$. It follows that the area of the half disk equals the area between the cycloid and the companion curve.

The companion curve is determined so that the height NP is the same as the height of F when the circle has rolled to base at N. It follows that the curve is symmetric with respect the center of $APXC$. If the circle starts at C and rolls left the same angle $CH = AF$ to the point Z, the corresponding height of H below the line DC equals ZX and the distance $WX = RP$, hence the length of PZ equals the length of XY. It follows that the area under the companion curve in the rectangle $ABCD$ equals the area above it, so is half the area of the rectangle $ABCD$.

It follows that the area $\mathcal{A}$ of half an arch of the cycloid equals the area between the cycloid and the companion curve plus the area under the companion curve

$$\mathcal{A} = \text{Area(Half Disk)} + \frac{1}{2}\,\text{Area(Rectangle } ABCD) = \frac{1}{2}\pi a^2 + \pi a^2 = \frac{3}{2}\pi a^2.$$

To check, lets do the usual area using calculus

$$\begin{aligned}\mathcal{A} &= \int_{t=0}^{\pi} y\,dx \\ &= a^2 \int_{t=0}^{\pi} (1-\cos t)^2 \; dt \\ &= a^2 \int_{t=0}^{\pi} 1 - 2\cos t + \cos^2 t\,dt \\ &= a^2 \left\{ \pi + 0 + \frac{1}{2}\pi \right\} = \frac{3}{2}\pi a^2.\end{aligned}$$

5. *Using Newton's version of Newton's method, find the root of the equation accurate to eight decimal places.*

$$f(y) = y^3 - y - 7 = 0$$

As a first guess, take $y_0 = 2$ because $f(2) = -1$ is close to zero. For the next step, add a correction and find the new resulting equation $y = 2 + p$.

$$0 = f(2+p) = 8 + 12p + 6p^2 + p^3 - 2 - p - 7 = -1 + 11p + 6p^2 + p^3.$$

Dropping the p^2 and p^3 terms because they are much smaller, we get

$$0 = -1 + 11p$$

so $p = .08$, approximately. (Actually, $p = .09$ gives a closer solution to the linearized equation, although this choice is better for the nonlinear equation. With this choice the next correction is positive. The procedure is self-correcting in that each iterate improves the estimate of the solution from whatever point you start.) Thus the second approximation is $y_1 = 2.08$. Substitute $p = .09 + q$ into the equation yields

$$\begin{aligned}0 &= -1 + 11p + 6p^2 + p^3 = -1 + 11(.08+q) + 6(.08+q)^2 + (.08+q)3 \\ &= -1 + 0.88 + 11q + 0.0384 + 0.96q + 6q^2 + 0.000512 + 0.0192q + 0.24q^2 + q^3 \\ &= -0.081088 + 11.9792q + 6.24q^2 + q^3\end{aligned}$$

Dropping q^2 and q^3 terms yields

$$0 = -0.081088 + 11.9792q$$

so $q = 0.0067$, approximately, so the corrected estimate is $y_2 = 2.0867$. Note that q is the the approximate correction os $y_1 -$ root, or the error made by the first iterate. Now substitute $q = 0.0067 + r$, we get

$$\begin{aligned}0 &= -0.081088 + 11.9792q + 6.24q^2 + q^3 \\ &= -0.081088 + 11.9792(0.0067 + r) + 6.24(0.0067 + r)^2 + (0.0067 + r)^3 \\ &= -0.081088 + 0.08026064 + 11.9792r + 0.0002801136 + 0.083616r + 6.24r^2 \\ &\qquad + 0.000000300763 + 0.00013467r + 0.0201r^2 + r^3 \\ &= -0.0005469456 + 12.06295r + 6.2601r^2 + r^3\end{aligned}$$

Neglecting r^2 and r^3 as before, we solve

$$0 = -0.0005469456 + 12.06295r$$

so $r = -0.00004534$, approximately. Thus the third approximate is $y_3 = 2.08674534$. Observing that $f(2.08674534) = 1.42 \times 10^{-9}$ and $f'(2.08674534) = 12.1$, the error y_3 makes is about $f(y_3)/f'(y_3) \approx 1.2 \times 10^{-10}$, which means that y_3 approximates the root to an error less that 0.5×10^{-8}, or to eight decimal places.

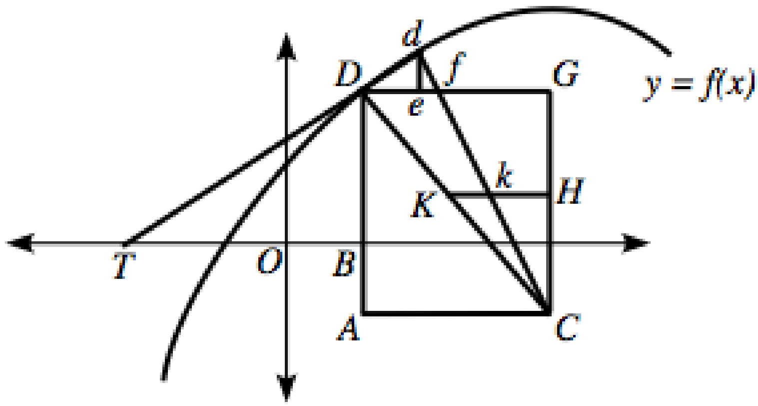
The modern way to compute Newton's method, which is equivalent to the above, is to make a good initial guess, y_0 and then to iterate

$$y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}.$$

We present a little **R**© program to compute the approximate y_i and correction p_i. The error in Newton's method is known to be the square of the previous error once the approximation is close, which is extremely fast. It gains about two decimal places each iteration.

```
> f<- function(x){x^3-x-7}
> fp <- function(x){3*x^2-1}
> y=2
> for( i in 1:5 ){ t = -f(y) / fp(y) ;
                  z = y + t ;
                  print( c(i,y,t), digits = 14 );
                  y = z }
[1]  1    2.0000000000000     0.090909090909091
[1]  2    2.0909090909091    -0.0041547810988465
[1]  3    2.0867543098102    -8.9698858243768e-06
[1]  4    2.0867453399244    -4.1753414965491e-11
[1]  5    2.0867453398827    -1.4725031199454e-16
```

6. *Following Newton, use the method of fluxions to find the curvature of the curve $y = \cosh x$.*



Let's determine the curvature at the point D on the curve $y = f(x)$. The tangent line passes through two infinitesimally close points D and d and T on the x-axis. The curvature κ is the reciprocal of the radius of the osculating circle, whose center C is the intersection point of the normal lines at D and d, thus it reflects the change of the slope of the tangent line from D to d. Thus $\frac{1}{\kappa} = DC$. Let $x = OB$ and $t = TO$ and $y = BD$ be fluents and $De = \dot{x}o$ and $de = \dot{y}o$ where $dotx$ and $\dot{y}$ are the fluxions (velocities) and o is an infintesimal time. It follows that the slope of the tangent line at D is

$$\frac{BD}{TB} = \frac{y}{x+t} = \frac{\dot{y}}{\dot{x}}.$$

Let $z = KH$ and $1 = CH$. Because TD is perpendicular to DC it follows that the triangles $\triangle Ded$, $\triangle CHK$ are similar. Thus it follows that

$$\frac{\dot{y}}{\dot{x}} = \frac{de}{De} = \frac{KH}{CH} = \frac{z}{1} = z.$$

Note how Newton has found a way to equate a ratio of velocities (derivaive) with a fluent in his diagram.

Let's solve for the length of DC using $\dot{z}o = Kk$, which is the derivative of $\frac{\dot{y}}{\dot{x}}$. Because we may consider $\triangle Ddf$ to be a right triangle, we have $\triangle Def$ is similar to $\triangle def$ so

$$\dot{y} = \frac{\dot{y}}{\dot{x}} = \frac{De}{de} = \frac{de}{ef}$$

so $ef = \frac{de^2}{De}$. On the other hand $Df = De + ef$ so

$$Df = De + \frac{de^2}{De} = \dot{x}o + \frac{\dot{y}^2 o}{\dot{x}}.$$

Finally, $\triangle CKk$ is similar to $\triangle CDf$ so

$$\dot{z}o = \frac{Kk}{CH} = \frac{Df}{CG}.$$

It follows that

$$CG = \frac{Df}{\dot{z}o} = \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\,\dot{z}}.$$

Also

$$DG = \frac{CG\,KH}{HC} = z\,CG$$

so if $\dot{x} = 1$ is steady horizontal motion,

$$DC = \sqrt{DG^2 + CG^2} = \frac{(1+z^2)^{\frac{3}{2}}}{\dot{z}}$$

Thus we may compute the curvature of $y = \cosh x$. Writing $z = \dot{y} = f'$,

$$\kappa = \frac{f''(x)}{(1+(f'(x))^2)^{\frac{3}{2}}} = \frac{\cosh x}{\left(1+\sinh^2 x\right)^{\frac{3}{2}}} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

7. *Use the binomial theorem to find a power series for* $f(x) = (1-x^2)^{\frac{1}{3}}$.

We compute a few binomial coeficients

$$\binom{\frac{1}{3}}{0} = 1; \quad \binom{\frac{1}{3}}{1} = \frac{\frac{1}{3}}{1} = \frac{1}{3}, \quad \binom{\frac{1}{3}}{2} = \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\cdot 2} = -\frac{1}{9}, \quad \binom{\frac{1}{3}}{3} = \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{1\cdot 2\cdot 3} = \frac{5}{81},$$
$$\binom{\frac{1}{3}}{4} = \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{1\cdot 2\cdot 3\cdot 4} = -\frac{10}{243}, \quad \binom{\frac{1}{3}}{4} = \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{22}{729}$$

Thus,

$$(1+t)^{\frac{1}{3}} = \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} t^k.$$

Setting $t = -x^2$ yields the desired series

$$(1-x^2)^{\frac{1}{3}} = 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6 - \frac{10}{243}x^8 - \frac{22}{729}x^{10} + \cdots$$

8. *Find an antiderivative of Newton's fluxional equation*

$$5x^3\dot{x} + 3x^2y^2\dot{x} + 2x^3y\dot{y} + 12y^5\dot{y} = 0$$

Newton replaces $x^p\dot{x}$ terms by $\dfrac{x^{p+1}}{p+1}$. Similar for $y^q\dot{y}$. Then he adds and removes any terms in common. Then checks. For the $\dot{x}$,

$$\frac{5}{4}x^4 + x^3y^2$$

and for the $\dot{y}$

$$x^3y^2 + 2y^6.$$

Removing one of the common terms after adding

$$\frac{5}{4}x^4 + x^3y^2 + 2y^6 = c$$

One checks that the implicit velocities are

$$5x^3\dot{x} + 3x^2y^2\dot{x} + 2x^3y\dot{y} + 12y^5\dot{y} = 0.$$

9. *Following Newton, find a series solution for $y(x)$.*

$$\dot{y}^2 + x\dot{x}\dot{y} - \dot{x}^2 = 0$$

Newton knew $\frac{\dot{y}}{\dot{x}} = x^n$ implies $y = \frac{y^{n+1}}{n+1} + C$. So in this case, we may solve for the ratio

$$\left(\frac{\dot{y}}{\dot{x}}\right)^2 + x\frac{\dot{y}}{\dot{x}} - 1 = 0$$

so from the quadratic formula

$$\frac{\dot{y}}{\dot{x}} = \frac{-x \pm \sqrt{x^2+4}}{2} = -\frac{x}{2} \pm \sqrt{\left(\frac{x}{2}\right)^2 + 1}$$

Computing a few binomial coefficients

$$\binom{\frac{1}{2}}{0} = 1; \quad \binom{\frac{1}{2}}{1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}, \quad \binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\cdot 2} = -\frac{1}{2^3}, \quad \binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot 2\cdot 3} = \frac{1}{2^4},$$
$$\binom{\frac{1}{2}}{4} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\cdot 2\cdot 3\cdot 4} = -\frac{5}{2^7}, \quad \binom{\frac{1}{2}}{5} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{7}{2^8}$$

so

$$\frac{\dot{y}}{\dot{x}} = -\frac{x}{2} \pm \left\{1 + \frac{1}{2^3}x^2 - \frac{1}{2^7}x^4 + \frac{1}{2^{10}}x^6 - \frac{5}{2^{15}}x^8 + \frac{7}{2^{18}}x^{10} + \cdots\right\}$$

which gives the two solutions

$$y = x - \frac{1}{4}x^2 + \frac{1}{3\cdot 2^3}x^3 - \frac{1}{5\cdot 2^7}x^5 + \frac{1}{7\cdot 2^{10}}x^7 - \frac{5}{9\cdot 2^{15}}x^9 + \frac{7}{2^{11\cdot 18}}x^{11} + \cdots$$
$$y = -x - \frac{1}{4}x^2 - \frac{1}{3\cdot 2^3}x^3 + \frac{1}{5\cdot 2^7}x^5 - \frac{1}{7\cdot 2^{10}}x^7 + \frac{5}{9\cdot 2^{15}}x^9 - \frac{7}{2^{11\cdot 18}}x^{11} + \cdots$$

10. *Find the sum of the series.*

$$S = \frac{13}{1\cdot 2\cdot 5\cdot 7} + \frac{25}{2\cdot 3\cdot 7\cdot 9} + \frac{41}{3\cdot 4\cdot 9\cdot 11} + \frac{61}{4\cdot 5\cdot 11\cdot 13} + \frac{85}{5\cdot 6\cdot 13\cdot 15} + \cdots$$
$$= \sum_{n=1}^{\infty} \frac{2n^2+6n+5}{n\cdot(n+1)\cdot(2n+3)\cdot(2n+5)}.$$

Some people failed to understand summation problem in the homework involving telescoping (collapsing) sums. The idea is to use a telescoping sum here. One notices that the general term is

$$x_n - x_{n+1} = \frac{n+1}{n(2n+3)} - \frac{(n+1)+1}{(n+1)(2(n+1)+1)} = \frac{2n^2+6n+5}{n\cdot(n+1)\cdot(2n+3)\cdot(2n+5)}.$$

The partial sum telescopes

$$\sum_{n=1}^{N} \frac{2n^2+6n+5}{n\cdot(n+1)\cdot(2n+3)\cdot(2n+5)} = \sum_{n=1}^{N}(x_n - x_{n+1})$$
$$= x_1 - x_{N+1} = \frac{2}{1\cdot 5} - \frac{(N+1)+1}{(N+1)(2(N+1)+1)}.$$

The infinite sum is the limit of the partial sums as $N \to \infty$, or

$$S = \frac{2}{1\cdot 5} - 0 = \frac{2}{5}.$$

11. *Let f_n be the nth Fibonacci number ($f_0 = 0,\ f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \ge 2$.) Show that*

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \frac{1}{1 + \dfrac{f_n}{f_{n+1}}} = \frac{f_{n+1}}{f_{n+2}}$$

and deduce a continued fraction for the golden mean $\tau = \dfrac{1+\sqrt{5}}{2}$. [Stillwell, Mathematics and its History, p. 195.]

Using the closed form expression of the Fibonacci numbers

$$f_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right\} = \frac{1}{\sqrt{5}} (\tau_+^n - \tau_-^n)$$

where $\tau_\pm = \dfrac{1 \pm \sqrt{5}}{2}$. Since $|\tau_-| < 1 < \tau_+$ we have

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \frac{\tau_+^{n+1} - \tau_-^{n+1}}{\tau_+^n - \tau_-^n} = \lim_{n\to\infty} \frac{\tau_+ - \tau_+^{-n}\tau_-^{n+1}}{1 - \tau_+^{-n}\tau_-^n} = \frac{\tau + 0}{1 + 0} = \tau.$$

Proving the limit statement. By the recursion,

$$\frac{f_{n+1}}{f_{n+2}} = \frac{f_{n+1}}{f_{n+1} + f_n} = \frac{1}{\dfrac{f_{n+1} + f_n}{f_{n+1}}} = \frac{1}{1 + \dfrac{f_n}{f_{n+1}}}$$

proving the second statement. We sequentially build up the continued fraction.

$$\begin{aligned}
\frac{f_3}{f_2} &= \frac{f_2 + f_1}{f_2} = 1 + \frac{f_1}{f_2} = 1 + 1 \\
\frac{f_4}{f_3} &= \frac{f_3 + f_2}{f_3} = 1 + \frac{f_2}{f_3} = 1 + \frac{1}{1 + \dfrac{f_1}{f_2}} = 1 + \frac{1}{1+1} \\
\frac{f_5}{f_4} &= \frac{f_4 + f_3}{f_4} = 1 + \frac{f_3}{f_4} = 1 + \frac{1}{1 + \dfrac{f_2}{f_3}} = 1 + \cfrac{1}{1 + \cfrac{1}{1+1}} \\
\frac{f_6}{f_5} &= \frac{f_5 + f_4}{f_5} = 1 + \frac{f_4}{f_5} = 1 + \frac{1}{1 + \dfrac{f_3}{f_4}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+1}}} \\
\frac{f_7}{f_6} &= \frac{f_6 + f_5}{f_6} = 1 + \frac{f_5}{f_6} = 1 + \frac{1}{1 + \dfrac{f_4}{f_5}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1+1}}}}
\end{aligned}$$

and so on. Finally

$$\frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{f_{n+1}}{f_n} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}$$

12. *Given that $P_n = n!$, deduce the following identities in Leibnitz's* Ars Combinatoria. *[Burton,* The History of Mathematics, *p. 432.]*

$$2P_n - (n-1)P_{n-1} = P_n + P_{n-1}, \qquad \text{and} \qquad P_n^2 = P_{n-1}(P_{n+1} - P_n)$$

An identity is proved by starting at one end and deducing a sequence of equal quantities until the other end is reached. It is not proved by assuming the identity and working from there. We use $n! = n(n-1)!$. To show the first identity.

$$\begin{aligned} 2P_n - (n-1)P_{n-1} &= 2n! - (n-1)(n-1)! \\ &= 2n(n-1)! - (n-1)(n-1)! \\ &= \big[2n - (n-1)\big](n-1)! \\ &= \big[n+1\big](n-1)! \\ &= n(n-1)! + (n-1)! \\ &= n! + (n-1)! \\ &= P_n + P_{n-1}. \end{aligned}$$

For the second,

$$\begin{aligned} P_n^2 &= n! \cdot n! \\ &= (n-1)! \cdot n \cdot n! \\ &= (n-1)!\,\big[(n+1) - 1\big]n! \\ &= (n-1)!\,\big[(n+1)n! - n!\big] \\ &= (n-1)!\big[(n+1)! - n!\big] \\ &= P_{n-1}\,\big[P_{n+1} - P_n\big]. \end{aligned}$$

13. *Use Euler's method to sum the series [Katz,* A History of Mathematics, *p. 638.]*

$$S = \sum_{k=1}^{\infty} \frac{1}{k^4}$$

Euler regards functions given by power series as extended polynomials. He employs the relation between the coefficients of the polynomial and the roots. Let $x_1, \ldots, x_n$ be the roots of the polynomial then

$$p(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \left(1 - \frac{x}{x_1}\right)\left(1 - \frac{x}{x_2}\right)\cdots\left(1 - \frac{x}{x_n}\right)$$

implies that

$$-a_1 = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$$

as one can see by multiplying the factors. Thus we need an analytic function whose zeros are k^4. The trick is to consider the sinc function

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} \cdots \tag{2}$$

whose zeros are $\pm\pi, \pm 2\pi, \pm 2\pi, \ldots$ but not 0 because sinc has the value one at zero. The corresponding product formula is

$$\begin{aligned} \frac{\sin x}{x} &= \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots \\ &= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots \end{aligned}$$

Replacing x by ix gives the series for $\sinh x = \dfrac{e^x - e^{-x}}{2}$. It has the power series

$$\frac{\sinh x}{x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} \cdots \tag{3}$$

The zeros are $\pm\pi i k$ so it has the product

$$\begin{aligned}\frac{\sinh x}{x} &= \left(1 - \frac{x}{\pi i}\right)\left(1 + \frac{x}{\pi i}\right)\left(1 - \frac{x}{2\pi i}\right)\left(1 + \frac{x}{2\pi i}\right)\left(1 - \frac{x}{3\pi i}\right)\left(1 + \frac{x}{3\pi i}\right)\cdots \\ &= \left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\left(1 + \frac{x^2}{3^2\pi^2}\right)\cdots\end{aligned}$$

Thus the product of these two has the product formula

$$\begin{aligned}\frac{\sinh x\, \sin x}{x^2} &= \left(1 + \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 + \frac{x^2}{3^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots \\ &= \left(1 - \frac{x^4}{\pi^4}\right)\left(1 - \frac{x^4}{2^4\pi^4}\right)\left(1 - \frac{x^4}{3^2\pi^4}\right)\cdots\end{aligned}$$

Thus this function's power series has powers of four, namely multiplying (2) and (3),

$$\begin{aligned}\frac{\sinh x\, \sin x}{x^2} &= 1 + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^2 + \left(\frac{1}{5!} - \frac{1}{(3!)^2} + \frac{1}{5!}\right)x^4 + \left(\frac{1}{7!} - \frac{1}{3!5!} + \frac{1}{3!5!} - \frac{1}{7!}\right)x^6 + \\ &\qquad + \left(\frac{1}{9!} - \frac{1}{3!\,7!} + \frac{1}{(5!)^2} - \frac{1}{3!\,7!} + \frac{1}{9!}\right)x^8 + \cdots \\ &= 1 - \frac{1}{90}x^4 + \frac{16}{5\cdot 9!}x^8 + \cdots\end{aligned}$$

Substituting $x^4 = y$, the function

$$\frac{\sinh \sqrt[4]{y}\, \sin \sqrt[4]{y}}{\sqrt{y}} = 1 - \frac{1}{90}y + \frac{16}{5\cdot 9!}y^2 + \cdots = \left(1 - \frac{y}{\pi^4}\right)\left(1 - \frac{y}{2^4\pi^4}\right)\left(1 - \frac{y}{3^2\pi^4}\right)\cdots$$

has the roots $y = \pi^4 k^4$ for $k = 1, 2, 3, \ldots$. The sum of the reciprocals is thus

$$\frac{1}{90} = \frac{1}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)$$

as desired. $S = \dfrac{\pi^4}{90}$.

14. *Find the generating function for the triangular numbers.*

For a sequence $\{a_i\}$, the generating function is given by the power series

$$f(x) = \sum_{k=1}^{\infty} a_k x^k$$

The triangular numbers are $1, 3, 6, 10, \ldots$, the number of dots in a triangle of side $1, 2, 3, 4, \ldots$. They are given by $1, 1+2, 1+2+3, 1+2+3+4, \ldots$, in other words

$$a_n = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

We note that

$$\Delta a_n = a_n - a_{n-1} = \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = n$$

and

$$\Delta^2 a_n = (a_n - a_{n-1}) - (a_{n-1} - a_{n-2}) = n - (n-1) = 1.$$

thus the triangular numbers satisfy the recursion

$$a_1 = 1, \qquad a_2 = 3, \qquad a_n - 2a_{n-1} + a_{n-2} = 1 \quad \text{for } n \geq 3.$$

So substituting $j = k$, $j = k+1$ and $j = k+2$,

$$\begin{aligned}
f(x) &= \sum_{k=1}^{\infty} a_k x^k & &= \sum_{j=1}^{\infty} a_j x^j \\
x f(x) &= \sum_{k=1}^{\infty} a_k x^{k+1} & &= \sum_{j=2}^{\infty} a_{j-1} x^j \\
x^2 f(x) &= \sum_{k=1}^{\infty} a_k x^{k+2} & &= \sum_{j=3}^{\infty} a_{j-2} x^j
\end{aligned}$$

It follows from

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

and the recursion that

$$\begin{aligned}
f(x) - 2xf(x) + x^2 f(x) &= \Big(a_1 x + a_2 x^2\Big) - 2\Big(a_1 x^2\Big) + \sum_{j=3}^{\infty} \Big(a_j - 2a_{j-1} a_{j-2}\Big) x^j \\
&= \Big(x + 3x^2\Big) - 2\Big(x^2\Big) + \sum_{j=3}^{\infty} x^j \\
&= x + x^2 + \left(\frac{1}{1-x} - 1 - x - x^2\right) \\
&= \frac{1}{1-x} - 1 = \frac{x}{1-x}
\end{aligned}$$

It follows that

$$(1 - 2x + x^2) f(x) = \frac{x}{1-x}$$

or

$$f(x) = \frac{x}{(1-x)^3}.$$

Let us derive the formula another way.

$$\begin{aligned}
f(x) &= \sum_{k=1}^{\infty} a_k x^k \\
&= \frac{1}{2}\sum_{k=1}^{\infty}(k+1)k\,x^k \\
&= \frac{x}{2}\sum_{k=1}^{\infty}(k+1)k\,x^{k-1} \\
&= \frac{x}{2}\frac{d}{dx}\left(\sum_{k=1}^{\infty}(k+1)\,x^k\right) \\
&= \frac{x}{2}\frac{d^2}{dx^2}\left(\sum_{k=1}^{\infty}x^{k+1}\right) \\
&= \frac{x}{2}\frac{d^2}{dx^2}\left(\frac{1}{1-x}-1-x\right) = \frac{x}{(x-1)^3}.
\end{aligned}$$

Here is a third way. Observe that

$$\begin{aligned}
&\left(1+x+x^2+x^3+\cdots\right)\left(a_1x+a_2x^2+a_3x^3+\cdots\right) = \\
&\qquad a_1x+(a_1+a_2)x^2+(a_1+a_2+a_3)x^3+\cdots
\end{aligned}$$

which says

$$\frac{f(x)}{1-x} = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}a_j\right)x^k$$

It follows that

$$\begin{aligned}
\frac{x}{(1-x)^3} &= \frac{x}{(1-x)^2}\sum_{k=0}^{\infty}x^k \\
&= \frac{1}{(1-x)^2}\sum_{k=1}^{\infty}x^k \\
&= \frac{1}{1-x}\sum_{k=1}^{\infty}k\,x^k \\
&= \sum_{k=1}^{\infty}\frac{k(k+1)}{2}x^k \\
&= \sum_{k=1}^{\infty}a_kx^k.
\end{aligned}$$

15. *Fermat considered the problem of how to divide the stakes of a two player game if the game is interrupted before all rounds have been played. Suppose both players I and II have an equal chance of winning each round. Suppose player I needs to win k of the next n rounds, where $n \geq k \geq 1$. Show that player I's share should be*

$$\frac{1}{2^n}\sum_{j=k}^{n}\binom{n}{j}$$

of the total stake. [Stillwell, Mathematics and its History, p. 207.]

Fermat's principle is that if a game should stop before the last play and there are two possibile outcomes, either I wins A on the last play or II wins B on the last play with both outcomes equally likely equally, then the share of the stake I should get is half of the total. Let us argue by induction on n. Thus if $k = 1$ and $n = 1$, then players I and II are equally likely to win and I's share should be half. But for $k = n = 1$,

$$\frac{1}{2^1}\sum_{j=1}^{1}\binom{1}{j} = \frac{1}{2}$$

which completes the base case. Now suppose the formula is correct for any $1 \le k \le n$. Suppose $n+1$ games are left and they play the next game. Two outcomes are possible: I loses and he needs k more wins or I wins and he needs $k-1$ more wins in the remaining n games. Thus we may apply the induction hypothesis: the proportion of I's wins should he the average of the two outcomes, namely,

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{2^n}\sum_{j=k}^{n}\binom{n}{j} + \frac{1}{2^n}\sum_{j=k-1}^{n}\binom{n}{j}\right) &= \frac{1}{2^{n+1}}\left(\sum_{j=k-1}^{n-1}\binom{n}{j+1} + \sum_{j=k-1}^{n-1}\binom{n}{j} + \binom{n}{n}\right)\\ &= \frac{1}{2^{n+1}}\left(\sum_{j=k-1}^{n-1}\binom{n+1}{j+1} + 1\right)\\ &= \frac{1}{2^{n+1}}\left(\sum_{j=k}^{n}\binom{n+1}{j} + \binom{n+1}{n+1}\right)\\ &= \frac{1}{2^{n+1}}\sum_{j=k}^{n+1}\binom{n+1}{j}\end{aligned}$$

so the induction is complete.

16. *Show*

$$\int_0^1 x^n(\log x)^n\,dx = \frac{(-1)^n n!}{(n+1)^{n+1}}.$$

Then prove Johann Bernoulli's formula. [Stillwell, Mathematics and its History, p. 274.]

$$\int_0^1 x^x\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$

First note that for integer $a \ge 1$, by L'Hopital's rule

$$\lim_{x\to 0+} x\log^a x = \lim_{x\to 0+}\frac{\log^a x}{\frac{1}{x}} = \lim_{x\to 0+}\frac{\frac{a\log^{a-1} x}{x}}{-\frac{1}{x^2}} = -a\lim_{x\to 0+} x\log^{a-1} x$$

so that repeating a times

$$\lim_{x\to 0+} x\log^a x = (-1)^a a!\lim_{x\to 0+} x = 0.$$

Thus the function $x^n(\log x)^n$ is continuous on $[0,1]$ so integrable. For integers $a, b \ge 1$, integration by parts gives $u = (\log x)^b$ and $dv = x^a\,dx$ so $du = \dfrac{b(\log x)^{b-1}\,dx}{x}$ and $v = \dfrac{x^{a+1}}{a+1}$

so

$$\int_0^1 x^a (\log x)^b \, dx = \left[\frac{(\log x)^b x^{a+1}}{a+1} \right]_0^1 - \frac{b}{a+1} \int_0^1 x^a (\log x)^{b-1} \, dx$$
$$= -\frac{b}{a+1} \int_0^1 x^a (\log x)^{b-1} \, dx$$

Applying this n times yields the first equation

$$\begin{aligned}
\int_0^1 x^n (\log x)^n \, dx &= -\frac{n}{n+1} \int_0^1 x^n (\log x)^{n-1} \, dx \\
&= \frac{n(n-1)}{(n+1)^2} \int_0^1 x^n (\log x)^{n-2} \, dx \\
&\vdots \\
&= \frac{(-1)^n n!}{(n+1)^n} \int_0^1 x^n \, dx \\
&= \frac{(-1)^n n!}{(n+1)^{n+1}}.
\end{aligned}$$

Now, using the fact that

$$x^x = e^{x \log x} = 1 + x \log x + \frac{x^2 (\log x)^2}{2!} + \frac{x^3 (\log x)^3}{3!} + \frac{x^4 (\log x)^4}{4!} + \cdots$$

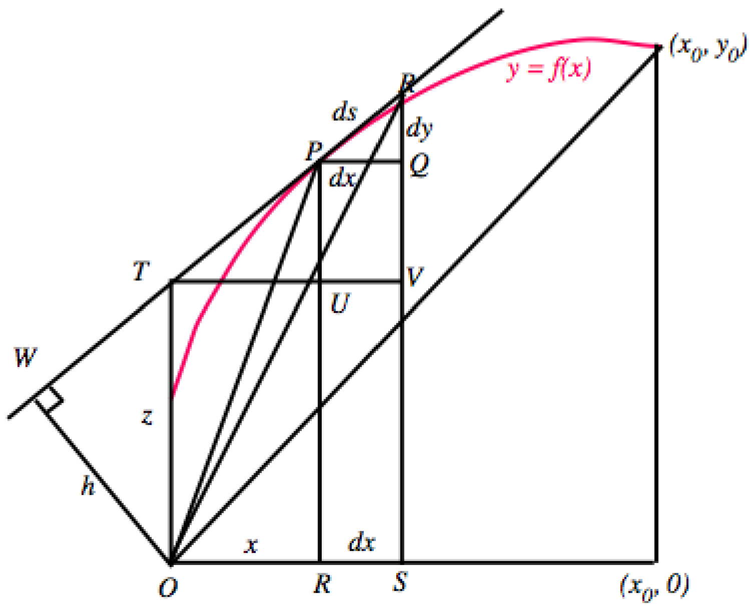
Integrating and using the integrals above gives $\int_0^1 x^x \, dx =$

$$\int_0^1 dx + \int_0^1 x \log x \, dx + \int_0^1 \frac{x^2 (\log x)^2}{2!} dx + \int_0^1 \frac{x^3 (\log x)^3}{3!} dx + \int_0^1 \frac{x^4 (\log x)^4}{4!} \, dx + \cdots$$
$$= 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$

17. *Use Leibnitz's Transmutation Formula to compute the integral, and check.*

$$\int_0^1 2x - x^2 \, dx$$

Of course, one would compute the integral directly, but this will show us what the transmutation formula looks like in this case.



The fundamental triangle is $\triangle PQR$ has $ds = \sqrt{dx^2 + dy^2}$. The slope of the tangent line is $\dfrac{dy}{dx}$. The distance of the tangent line to the otigin and is $h = OW$ and its y-intercept is $z = OT$. Since $\triangle PQR$ is similar to $\triangle OWT$ we have

$$\frac{ds}{dx} = \frac{z}{h}$$

or $h\,ds = z\,dx$. The area of the triangle $\triangle OPQ$ is half the base times height or $\frac{1}{2}h\,ds$. thus the total area under the curve is

$$\int_0^{x_0} y\,dx = \frac{1}{2}\int_0^{x_0} h\,ds + \frac{1}{2}x_0y_0$$

where the integral on the right is the area of the region below the curve and above the line from O to (x_0, y_0). The second term is the area of the triangle from O to $(x_0, 0)$ to (x_0, y_0). Using $h\,ds = z\,dx$ we get the transmutation formula

$$\int_0^{x_0} y\,dx = \frac{1}{2}\int_0^{x_0} z\,dx + \frac{1}{2}x_0y_0$$

For the integral given by the problem, $y = f(x) = 2x - x^2$, $(x_0, y_0) = (1, 1)$ and the slope of the tangent line is

$$m = \frac{dy}{dx} = 2 - 2x.$$

Thus using the point-slope form, at pont (x, y) and slope m, the equation of the tangent line is

$$Y - y = (2 - 2x)(X - x)$$

The tangent line crosses the y-axis at $(X, Y) = (0, z)$, so

$$z = y - (2 - 2x)x = 2x - x^2 - 2x + 2x^2 = x^2$$

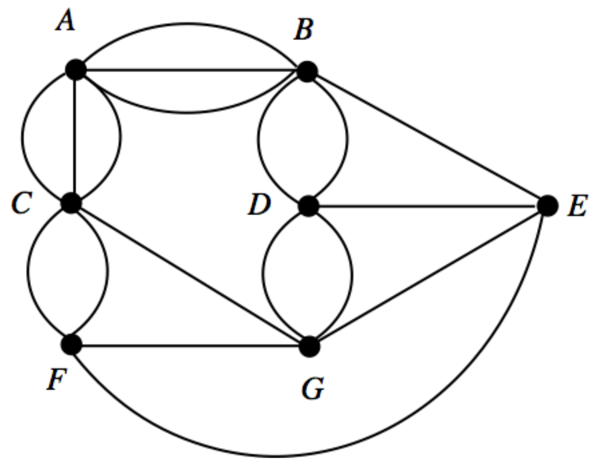
The transmutation formula is then

$$\int_0^1 2x - x^2\,dx = \int_0^{x_0} y\,dx = \frac{1}{2}\int_0^{x_0} z\,dx + \frac{1}{2}x_0y_0 = \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{2}.$$

Now both sides give the same value.

$$\int_0^1 2x - x^2\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3}, \qquad \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{2} = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2} = \frac{1}{6} + \frac{1}{2}.$$

18. *Determine whether there is an Eulerian path through the following graph. If ther is one, what is it?*



An Eulerian path is a path starting at one node that crosses each edge exactly once. The valences (number of edges at a vertex) $v_A = 6$, $v_B = 6$, $v_C = 6$, $v_D = 5$, $v_E = 4$, $v_F = 4$ and $v_G = 5$. There are all together $n = 18$ edges. Euler gave the following formula that has to hold if and only if there is an Eulerian path. At each node (region) compute the number of passages

$$\ell = \begin{cases} \dfrac{v+1}{2}, & \text{if } v \text{ is odd} \\[2ex] \dfrac{v}{2}, & \text{if } v \text{ is even and path does not start or end at the node} \\[2ex] \dfrac{v}{2} + 1, & \text{if } v \text{ is even and path starts and ends at the node.} \end{cases}$$

Then there is an Eulerian path if and only if the sum over all nodes

$$\ell_A + \cdots + \ell_G = \ n \text{ or } n+1.$$

In case there are more than two nodes with odd valence, this cannot hold. In our graph, there are two nodes with odd valence. They will be the start and endpoint of the Eulerian path. Ths means that for even valence, $\ell = \dfrac{v}{2}$. Computing,

$$\ell_A + \cdots + \ell_G = 3 + 3 + 3 + 3 + 2 + 2 + 3 = 19 = n + 1$$

Thus the Eulerian path is possible. One such path is $DBDEBABACACFCGFEGDG$.