Math 3010 § 1. **First Midterm Exam** Name: Solutions

Treibergs September 21, 2022

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1. (a) *Describe the style and contents of Euclid's Elements.*

   Elements is a riogorous text written in definition / axiom / proof style. It contains systematic developments of geometry and other Greek mathematics and records discoveries of Thales, Pythagoras, Eudoxus and others.

   (b) *For each mathematician fill a number for their principal location and a letter for their mathematical contribution.*

Mathematician	Location	Contribution
Thales 625 – 547 BC	6	A
Pythagoras 580 – 497 BC	4	E
Zeno 490 – 425 BC	5	F
Eudoxus 400 – 347 BC	3	D
Aristotle 384 – 322 BC	2	B
Diophantus 210 – 260 AD	1	C

Locations	**Mathematical Contributions**
1. Alexandria	A. Advocated the deductive method. First man to have a theorem attributed to him.
2. Athens	B. Advocated use of definition/axiom/proof in mathematical writing and syllogism logic.
3. Cnidus	C. Developed algebraic notation. Studied equations with integer unknowns.
4. Croton	D. Developed theories of proportion and exhaustion.
5. Elia	E. Explained musical harmony in terms of whole number ratios. Found that some lengths are irrational.
6. Miletus	F. Proposed paradoxes that conflicted with intuitive concepts of infinity.

2. *Use the Euclidean algorithm to find the greatest common divisor of 600 and 220. Find two integers x and y so that* $\gcd(600, 220) = 600x + 220y$. *Find another solution besides the one you gave.*

   Running the Euclidean Algorithm we find

$$
\begin{aligned}
600 &= 220 \cdot 2 + 160 \\
220 &= 160 \cdot 1 + 60 \\
160 &= 60 \cdot 2 + 40 \\
60 &= 40 \cdot 1 + 20 \\
40 &= 20 \cdot 2 + 0.
\end{aligned}
$$

   Hence $\gcd(600, 220) = 20$. Running the algorithm backwards we find

$$
\begin{aligned}
20 &= 60 - 40 & & \\
&= 60 - (160 - 60 \cdot 2) & &= 60 \cdot 3 - 160 \\
&= (220 - 160) \cdot 3 - 160 & &= 220 \cdot 3 - 160 \cdot 4 \\
&= 220 \cdot 3 - (600 - 220 \cdot 2) \cdot 4 & &= 220 \cdot 11 - 600 \cdot 4.
\end{aligned}
$$

   Thus $x = -4$ and $y = 11$. Check:

$$600 \cdot (-4) + 220 \cdot 11 = -2400 + 2420 = 20.$$

   Dividing by equation by gcd yields

$$30x + 11y = 1.$$

   Thus all solutions have the form

$$\tilde{x} = -4 - 11t, \qquad \tilde{y} = 11 + 30t,$$

   where t is an integer. For example, if $t = -1$ then $\tilde{x} = 7$ and $\tilde{y} = -19$. Check:

$$600 \cdot (7) + 220 \cdot (-19) = 4200 - 4180 = 20.$$

3. *Determine whether the following statements are true or false.*

   (a) *The Babylonians used a base 60 number system.*
   True.

   (b) *The Egyptians could calculate the area of a circle of diameter d.*
   True.

   (c) *Pythagoras discovered the Pythagorean Theorem.*
   False.

   (d) *Euclid gave a compass and straightedge construction of the regular pentagon.*
   True.

   Give a detailed explanantion of ONE of your answers (a)–(d) above.

   (a) The Old Babylonian numeration system is a base sixty place value system. Archeologists have found clay tablets with arithmetic tables used to facilitate computation.

   (b) The Egyptians used the formula $A = \left(\frac{8}{9}d\right)^2$ where d is the diameter of the circle. Thus they used the approximate value of $\pi \approx \frac{256}{81} = 3.1605$.

(c) The Babylonians were using the Pythagorean Theorem for geometric problems 1000 years before Pythagoras.

(d) The construction of the regular pentagon is in Book IV of the Euclid's Elements. It is essentially the same as described in your homework problem B3.

4. (a) *Use Pythagoras method to show* $\sqrt{7}$ *is irrational.*

Argue by contradiction. Assume that $\sqrt{7} = \frac{p}{q}$ is rational in lowest terms, where p and q are integers that have no common factors. Squaring yields

$$7q^2 = p^2.$$

This says that p^2 is divisible by seven. Since 7 is prime, then 7 must divide p. But this says that there is an integer k such that

$$p = 7k.$$

But this says

$$7q^2 = 7^2k^2,$$

or

$$q^2 = 7k^2.$$

This says that q^2 is divisible by 7. Again, since 7 is prime, this says that q must be divisible by seven also. But we have reached a contradiction since we have shown that both p and q are divisible by seven, and thus have a common factor, contrary to the choice of p and q. It follows that the contradiction hypothesis that $\sqrt{7}$ is rational is false: $\sqrt{7}$ is irrational.

(b) *Suppose you wished to show that another quantity a was equal to $\sqrt{7}$. According to Eudoxus, what do you need to establish to deduce $a = \sqrt{7}$?*

To show that a quantity a is equal to 7, Eudoxus method of proportion says that it is sufficient to prove first that if r were a rational number such that $\sqrt{7} < r$ then it must be that $a < r$ and, second, if t were a reational number such that $t < \sqrt{7}$ then it must be that $t < a$.

5. (a) [9] There are three famous Greek problems, the "Delian Problems." What are they? Could the Greeks solve them?

The Delian Problems are using just straightedge and compass, to square the circle, to double the cube and to trisect an angle. The Greeks found methods to solve all three, but by using methods other than just straightedge and compass. In the last century, mathematicians have shown that none of these problems may be solved using only a straightedge and compass.

(b) *Using a straightedge and compass, show how to bisect the angle $\angle AOB$. Explain your steps.*

Centering the compass point at the origin O, draw a (blue) circular arc AB passing through both legs of the angle so OA and OB have the same distance. Choose a sufficiently large radius and draw two (green) circles of that radius centered at A and B so they intersect at a point C inside the angle. Thus $AC = BC$. The line OC is the bisector of the angle. This is easily seen because the triangles $\Delta(AOC)$ and $\Delta(BOC)$ are congruent by SSS ($OA = OB$, $AC = BC$ and $OC = OC$.) Therefore the angles $\angle AOC$ and $\angle BOC$ are equal.