Homework for Math 3010 §1, Spring 2018

A. Treibergs, Instructor

November 17, 2022

Our text is by John Stillwell, *Mathematics and its History*, 3rd. ed., Springer, New York, 2010. Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Friday, or on April 24, whichever comes first.

Your written work reflects your professionalism. Make answers complete, self contained and written in good English. This means that you should copy or paraphrase each question, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Homework from Wednesday to Monday will be due Friday. Late homework that is up to one week late will receive half credit. Homework that is more than one week late will receive no credit at all. Homework that is placed in my mailbox in JWB 228 before 5:00 pm Friday afternoon will be considered to be on time.

Please hand in on paper problems A1 – A3 on Friday, August 26.

A1. Exercises from from Stillwell's *Mathematics and its History.*

1.2.3 Show that any integer square leaves remainder 0 or 1 on division by 4.

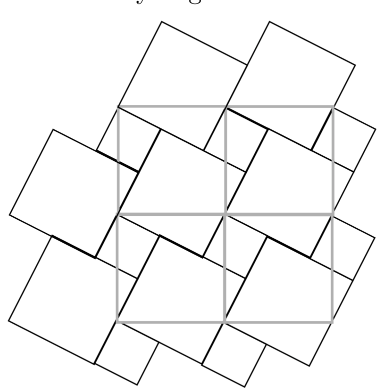
1.2.4 Deduce from [**1.2.3**] that if (a, b, c) is any Pythagorean triple then a and b cannot both be odd.

1.3.1 Deduce that if (a, b, c) is any Pythagorean triple then

$$\frac{a}{c} = \frac{p^2 - q^2}{p^2 + q^2}, \qquad \frac{b}{c} = \frac{2pq}{p^2 + q^2}$$

for some integers p, q. This shows that Pythagorean Triples are given by the formula in the middle of page 4. [Hint: This question asks you to give an argument for these expressions, say by solving the equation in the middle of page 8. It does not ask you to verify that these expressions satisfy $x^2 + y^2 = 1$.]

1.4.1 What has this figure to do with the Pythagorean Theorem?

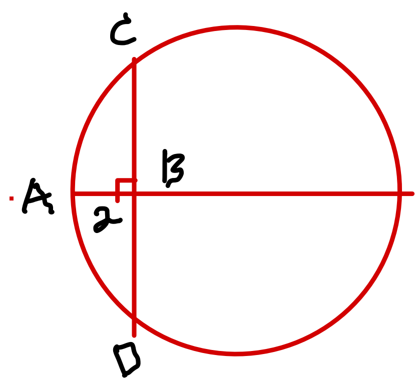


A2. From Bunt *et. al.* 4[4].

1. Use the ancient Egyptian procedure to find the area of a circle with diameter 12.
2. What would the area be if the modern value of π were used instead?
3. What percentage error made by the ancient Egyptians?

A3. From Bunt *et. al.* 63[8].

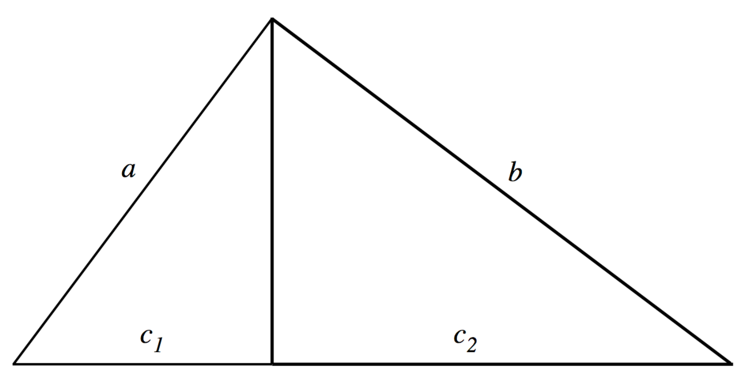
Solve the following problem that occurs on a Babylonian tablet. Given that the circumference of a circle is 60 and the length of the saggita $\overline{AB}$ is 2, calculate the length of the chord $\overline{CD}$ in the figure.



Please hand in problems B1–B3 on Friday, September 2.

B1. Please hand in the following exercises from from Stillwell's *Mathematics and its History.*

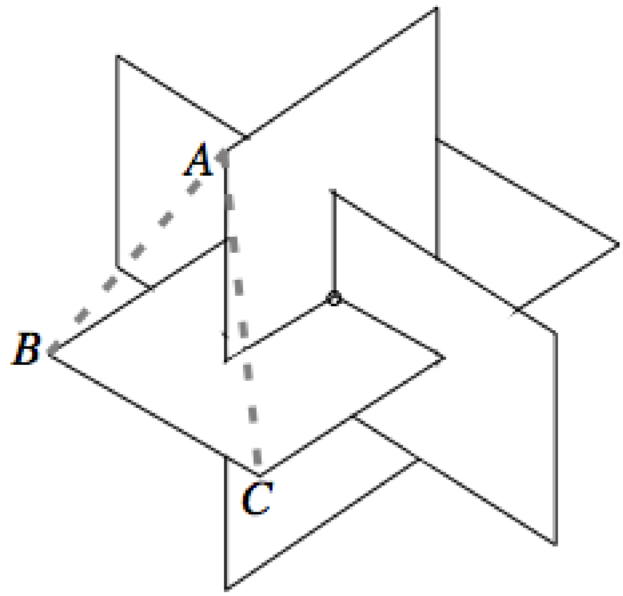
1.4.2 In Book VI of *Elements*, Euclid gives the following argument for the Pythagorean Theorem based on similar triangles. Show that the three triangles in the figure are similar, and hence prove the Pythagorean theorem by equating ratios of corresponding sides.



2.2.1 Show that for both cube and octohedron,

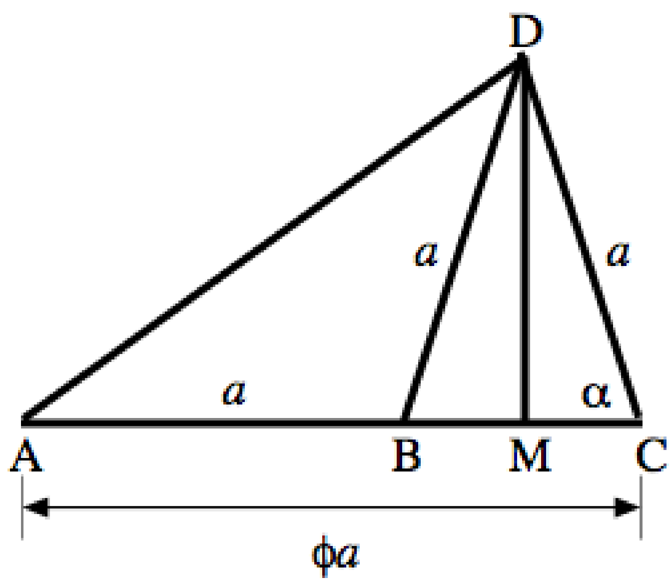
$$\frac{\text{circumradius}}{\text{inradius}} = \sqrt{3}$$

2.2.2 Check Pacioli's construction of the regular isosahedron. Consider the rectangle $[-\phi, \phi] \times [-1, 1]$. Suppose that three such rectangles are centered in the coordinate planes oriented so that the long directions align with the three axes as in the figure. Show that $AB = AC = BC$. Recall that $\phi = \frac{1}{2} + \frac{\sqrt{5}}{2}$ satisfies $\phi^2 = 1 + \phi$.



B2. Show that the Golden section ϕ is irrational.

B3. Check Euclid's construction of the regular pentagon. Suppose that collinear points A, B, C have distances $a = AB$, $\phi a = AC$. Construct the isosceles triangle $\triangle(BDC)$ with $a = BD = CD$. Let M be the bisector between B and C. Let the angle $\alpha = \angle(ACD)$.



a. Find the length of DM using the triangle $\triangle(BMD)$

b. Show that $AD = AC$ using the triangle $\triangle(AMD)$.

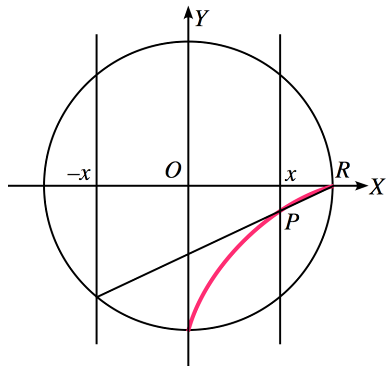
c. Express the angles of triangle $\triangle(ABD)$ in terms of α and show that $\alpha = 72°$. Thus it is the central angle of a sector of a regular pentagon. HINT: The sum of the interior angles of any triangle is $180°$.

Please hand in problems C1–C3 on Friday, Sept. 9.

C1. Please hand in the following exercises from from Stillwell's *Mathematics and its History.*

2.5.1 Using X and Y for the horizontal and vertical components, show that the straight line RP in Figure 2.10 of the text has equation

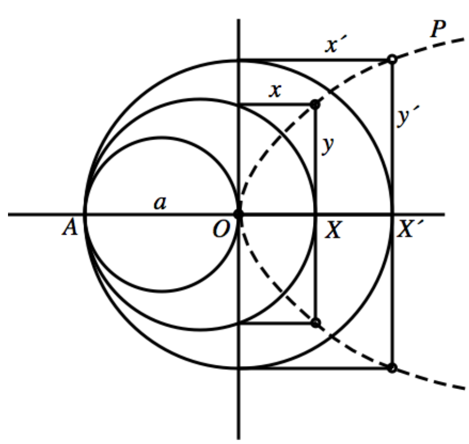
$$Y = \frac{\sqrt{1-x^2}}{1+x}(X-1).$$



2.5.2 Deduce the equation of the Cissoid of Diocles from Exercise 2.5.1.

C2. Given a line in the plane and a point P not on the line, describe how to construct a new line through P which is parallel to the first line by using straightedge and compass. Explain why your construction does the job.

C3. Here is how Menaechmus constructed the parabola as the locus of points $\mathcal{P} = \{(x, y)\}$. Given points A, O and X on line such that a is the distance AO and x is the distance OX. Let L be a perpendicular line through O. For each x, construct a circle whose diameter is AX. Then y is the distance from O to the intersection point of L with the circle. Show that Menaechmus' consruction yields the parabola. Show also that the parabola passes through the point (a, a).



Please hand in problems D1–D4 on Friday, September 16.

D1. Using the Euclidean algorithm, find $\gcd(963, 657)$.

D2. Find integers x and y that satisfy

$$\gcd(24, 138) = 24x + 138y.$$

D3. Show that if $a \mid bc$ and $\gcd(a, b) = 1$ then $a \mid c$.

D4. Prove that there are infinitely many primes of the form $6n - 1$. Suggestion: Use the number $P - 1$ where P is the product of all primes $\leq 6n - 1$.

D5. Please hand in the following exercises from from Stillwell's *Mathematics and its History.*

3.3.2 Show that for any integers a and b, there are integers m and n so that $\gcd(a, b) = ma + nb$.

3.3.3 Deduce from problem 3.3.2 that the equation $ax + by = c$ with integer coefficients has an integer solution x, y if $\gcd(a, b)$ divides c.

4.4,4 The equation $12x + 15y = 1$ has no integer solution. Why?

D6. On a separate piece of paper, write your Essay on Mathematics of Antiquity proposal. After the proposal is returned to you, please hand your proposal in again when you hand in your essay next week. Be sure to include in your proposal

- Working Title
- Short but specific description of what your essay is about. Don't just say you will discuss what the Greeks thought about π. Better say that you will describe how Archimedes showed that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. Everyone in class should have a different topic.
- State an interesting fact you've discovered about your topic in your preliminary readings.
- State which style manual you'll follow. You can find a list at the Mariott website `http://campusguides.lib.utah.edu/style`
- Give two internet references. Please include the author and the URL.
- Give two book or journal references specific to your topic (other than Stillwell).

Please hand in your essay E1 on Friday, Sept. 23.

E1. Essay on the Mathematics of Antiquity. Write an essay about a specific mathematical discovery/ theorem/ method that occurred before Christ.

- The paper should be five pages (in some reasonable font and font size) double-spaced and printed out on paper. It should be in written in good technical English. It should be written for an audience of Math 3010 students.
- There must be some mathematics, and mathematical explanation, in your paper. Just how you incorporate some mathematical exposition will vary from subject to subject. Include displayed equations and diagrams if appropriate.

- You must draw on a bare minimum of three book and journal sources. It is good if you include "primary" sources, quoting directly from the mathematician you're discussing, or at least from sources closest to them reconstructing the original source. A "Secondary" source is a scholarly interpretation later than the original subject of study in a book or journal. You may use blogs and Wiki articles provided that you give them credit. But also track down the source cited in a Wikipedia article.
- Give credit where it is due: whenever you use another author's ideas, whether appearing in your paper as direct quotation, paraphrase, or simply influence, you must cite them (with a footnote and then include in the bibliography). Formatting these citations and bibliography entries should be unambiguous, according to your chosen style guide. (Parts of these instructions are quoted from Patrikis's assignment 2-19-16.)
- Please attach your essay proposal from last week to your paper.

Please hand in problems F1–F6 on Friday, September 30.

F1. Find the least common multiple of 2646 and 8232.

F2. Use the Chinese square root algorithm to find $\sqrt{142,884}$.

F3. Use the Chinese cube root algorithm to find $\sqrt[3]{12,812,904}$.

F4. Solve Problem 1 from Chapter VII of *Nine Chapters*. Several people purchase in common one item. If each person paid 8 coins, the surplus is 3. If each paid 7, the deficiency is 4. How many people were there and what is the price of the item? [Katz, *A History of Mathematics,* 2009, p. 227]

F5. Solve Problem 3 from Chapter VIII of *Nine Chapters* using the Chinese method. None of the yields of two bundles of best grain, three bundles of ordinary grain and 4 bundles of the worst grain are sufficient to make a whole measure. If we add to the good grain 1 bundle of the ordinary, to the ordinary 1 bundle of the worst, and to the worst bundles 1 bundle of the best, then each yield is exactly one measure. How many measures does 1 bundle of each of the three types of grain contain? Show that the solution, according to the Chinese method involves the use of negative numbers. [Katz, p. 227]

F6. Solve Problem 20 from Chapter IX of *Nine Chapters*. A square walled city of unknown dimension has four gates at the centers of each side. A tree stands 20 *pu* north of the north gate. One must walk 14 *pu* south of the south gate and then turn west and walk 1775 *pu* before one can see the tree. What are the dimensions of the city? [Katz, p. 227]

Please hand in problems G1–G5 on Friday, October 7.

G1. Please hand in the following exercises from from Stillwell's *Mathematics and its History.*

6.2.1 Derive an equation that is linear in y from the two equations

$$\begin{aligned} x^2 + xy + y^2 &= 1 \\ 4x^2 + 3xy + 2y^2 &= 3, \end{aligned}$$

and hence show that $y = (1 - 2x^2)/x$.

6.2.2 Deduce that the intersection of the two curves in Exercise 6.2.1 occur where x satisfies $3x^4 - 4x^2 + 1 = 0$.

6.2.3 Solve $3z^2 - 4z + 1 = 0$ for $z = x^2$ by factoring the left-hand side, and hence find four solutions for x.

Give geometric reasons why you would expect two curves of degree 2 to have up to four intersections. Could they have more than four?

G2. Does the equation have an integral solution? If so, find all solutions

$$5049x + 2703y = 102$$

G3. Find the smallest positive solution of

$$300x \equiv 30 \mod 910.$$

G4. Find two positive x's such that all congruences hold

$$\begin{aligned} x &\equiv 2 \mod 3 \\ x &\equiv 3 \mod 5 \\ x &\equiv 4 \mod 7 \end{aligned}$$

G5. Solve problem 4 in section 1 in Qin Jiushao's *Treatise in Nine Sections.* Find N so that all five congruences hold.

$$\begin{aligned} N &\equiv 0 \mod 11 \\ N &\equiv 0 \mod 5 \\ N &\equiv 4 \mod 9 \\ N &\equiv 6 \mod 8 \\ N &\equiv 0 \mod 7 \end{aligned}$$

Please hand in problems H1–H3 on Friday, October 22.

H1. Please hand in the following exercises from from Stillwell's *Mathematics and its History.*

5.4.2 Derive Brahmagupta's identity using the factorization

$$(x_1^2 - Ny_1^2)(x_2^2 - Ny_2^2) = (x_1 - \sqrt{N}y_1)(x_1 + \sqrt{N}y_1)(x_2 - \sqrt{N}y_2)(x_1 + \sqrt{N}y_2)$$

and combining the first factor with the third and the second factor with the fourth.

5.6.1 From Brahmagupta: show that the triangle with sides 13, 14 15 splits as two integer right angled triangles.

6.6.2 Show that for *any* triangle with sides a, b, c and altitude h, there are real numbers u, v and w such that

$$a = \frac{u^2 + v^2}{v}, \qquad b = \frac{u^2 + w^2}{w}, \qquad c = \frac{u^2 - v^2}{v} + \frac{u^2 - w^2}{w},$$

with the side c split into parts $\dfrac{u^2 - v^2}{v}$ and $\dfrac{u^2 - w^2}{w}$ by the altitude $h = 2u$.

5.6.3 Define the *semiperimeter* of a triangle with sides a, b, c to be $s = \dfrac{a + b + c}{2}$. Then with the notation of Exercise 5.6.2, show that

$$s(s - a)(s - b)(s - c) = u^2(v + w)^2 \left(\frac{u^2}{vw} - 1\right)^2.$$

5.6.4 Deduce from Exercise 5.6.3 that

$$\sqrt{s(s-a)(s-b)(s-c)} = u\left(\frac{u^2-v^2}{v} + \frac{u^2-w^2}{w}\right)$$

is the area of a triangle with sides a, b and c. (This is Heron's formula.)

H2. Recall that for fixed N we write (a, b, k) whenever $a^2 - Nb^2 = k$. Here is another theorem of Brahmagupta.

Theorem. Assume $(x, y, 4)$. If x is odd or y is even, then $\left(x\left(\frac{x^2-3}{2}\right), y\left(\frac{x^2-1}{2}\right), 1\right)$. If x is even and y is odd then $\left(\frac{2x^2-4}{4}, \frac{xy}{2}, 1\right)$.

Check the theorem. Show that these triples give integral solutions and that they satisfy Pell's equation.

H3. Find a solution using Bhaskara II's method.

$$x^2 - 83y^2 = 1$$

Please hand in problems I1–I7 on Friday, October 28.

I1. Use Cardano's formula to solve these equations and check your solution by substitution.

1. $y^3 = 2$.
2. $y^3 = 6y + 6$
3. $x^3 + 3x^2 - 12x - 64 = 0$

I2. Use Brahmagupta's quadratic interpolation formula to find the Indian value for $\mathrm{Sin}(1000')$, where $R = 3438$.

Minutes	Sine	Sine Difference
0	0	* * *
225	225	225
450	449	224
675	671	222
900	890	219
1125	1105	215
1350	1315	210

I3. Find a pair of amicable numbers other than $(220, 284)$ using Thabit ibn Qurra's theorem.

I4. Show that 1184 and 1210 are amicable numbers that are not a consequence of the theorem of Thabit ibn Qurra.

I5. Show the result of Sharif al-Din (d. 1213) that one can solve

$$x^3 + d = cx$$

by intersecting the hyperbola and the parabola

$$y^2 - x^2 + \frac{d}{c}x = 0,$$
$$x^2 = \sqrt{c}\,y.$$

Sketch the two conics. Find sets of values for c and d for which these conics do not intersect, intersect once, and intersect twice.

I6. How many different 13 letter words can be made out of an alphabet with 26 letters such that there are five non-repeating letters, two letters that repeat twice and one letter that repeats four times?

I7. On a separate piece of paper, write your Essay on Renaissance Mathematics proposal. It should be about a specific mathematical discovery/ theorem/ method that occurred between AD 500 – AD 1850. After the proposal is returned to you, please hand your proposal in again when you hand in your essay next week. Be sure to include in your proposal

- Working Title
- Short but specific description of what your essay is about. Don't just say you will discuss what the Omar Khayyam thought about Euclid. Better say you will describe Khayyam's suggestions to replace for the parallel postulate which anticipate the modern development of non-Euclidean geometry. Everyone in class should have a different topic.
- State an interesting fact you've discovered about your topic in your preliminary readings.
- State which style manual you'll follow. You can find a list at the Mariott website `http://campusguides.lib.utah.edu/style`
- Give two internet references. Please include the author and the URL.
- Give two book or journal references specific to your topic (other than Stillwell). One of your sources must be a primary source, quoting directly from the mathematician being discussed.

Please hand in your essay J1 on Friday, Nov. 4.

J1. Essay on the Renaissance Mathematics. Write an essay about a specific mathematical discovery/ theorem/ method that occurred between AD 500 – AD 1850.

- The paper should be five pages (in some reasonable font and font size) double-spaced and printed out on paper. It should be in written in good technical English. It should be written for an audience of Math 3010 students.
- There must be some mathematics, and mathematical explanation, in your paper. Just how you incorporate some mathematical exposition will vary from subject to subject. Include displayed equations and diagrams if appropriate.
- You must draw on a bare minimum of three book and journal sources. ONE MUST BE a "primary" source, quoting directly from the mathematician you're discussing, or at least from sources closest to them reconstructing the original source. A "Secondary" source is a scholarly interpretation later than the original subject of study in a book or journal. You may use blogs and Wiki articles provided that you give them credit. But also track down the source cited in a Wikipedia article.

- Give credit where it is due: whenever you use another author's ideas, whether appearing in your paper as direct quotation, paraphrase, or simply influence, you must cite them (with a footnote and then include in the bibliography). Formatting these citations and bibliography entries should be unambiguous, according to your chosen style guide. (Parts of these instructions are quoted from Patrikis's assignment 2-19-16.)
- Please attach your essay proposal from last week to your paper.

Please hand in problems K1 – K2 on Friday, November 11.

K1. Please hand in the following exercises from from Stillwell's *Mathematics and its History.*

7.1.1 Generalize the idea of Menaechmus to show that any cubic equation

$$ax^3 + bx^2 + cx + d = 0 \qquad \text{with } d \neq 0$$

may be solved by intersecting a hyperbola $xy = 1$ with a parabola.

7.2.1 Show that a quadratic form $ax^2 + bxy + cy^2$ may be converted to a form $\tilde{a}\tilde{x}^2 + \tilde{b}\tilde{y}^2$ by suitable choice of θ in the substitution

$$\begin{aligned} x &= \tilde{x}\cos\theta - \tilde{y}\sin\theta \\ y &= \tilde{x}\sin\theta + \tilde{y}\cos\theta, \end{aligned}$$

by checking the that the coefficient of $\tilde{x}\tilde{y}$ is $(c-a)\sin 2\theta + b\cos 2\theta$ and that for appropriate θ this may be made to vanish.

7.4.2 Find rational functions $x = r(t)$ and $y = s(t)$ that parameterize

$$y^2 = x^2(x+1)$$

by finding the second intersection of the line $y = tx$ through the double point of the curve.

7.5.2 Find two parabolas whose intersections give the solutions of

$$x^4 = x + 1,$$

and hence show that this quartic equation has two real roots.

K2. Recall Napier's logarithm $\text{Nlog}(x) = m$ if $10^7(1 - 10^{-7})^m = x$. Show that

$$\text{Nlog}(x) + \text{Nlog}(y) = \text{Nlog}(xy) + \text{Nlog}(1)$$

Please hand in problems L1 on Friday, April 6.

L1. Please hand in the following exercises from from Stillwell's *Mathematics and its History.*

9.2.1 Find $1 + 2 + \cdots + n$ by summing the identitiy

$$(m+1)^2 - m^2 = 2m + 1$$

from $m = 1$ to n. Similarly find $1^2 + 2^2 + \cdots + n^2$ using the identity

$$(m+1)^3 - m^3 = 3m^2 + 3m + 1$$

together with the previous result. Likewise, find $1^3 + 2^3 + \cdots + n^3$ using the identity

$$(m+1)^4 - m^4 = 4m^3 + 6m^2 + 4m + 1.$$

9.2.2 Show that the approximation to the area under $y = x^2$ for $0 \le x \le 1$ by rectangles as in Figure 9.1 has the value $(2n+1)n(n+1)/6n^3$, and deduce the area under the curve is $1/3$.

9.2.4 Use Cavalieri's method of indivisibles to prove Archimedes' formula for the volume of a sphere as follows. Show that the slice $z = c$ of the sphere $x^2 + y^2 + z^2 \le 1$ has the same area as the slice $z = c$ of the cylinder $x^2 + y^2 \le 1$ outside the cone $x^2 + y^2 = z^2$.

9.2.5 Deduce from exercise 9.2.4, and the known volume of the cone, that the volume of the sphere is 2/3 the volume of the circumscribing cylinder.

9.3.2 For evidence that tangents to algebraic curves may be found without calculus, it is enough to look at Diophantus's tangent method of section 3.5. Diophantus finds the tangent $y = \frac{3}{2}x+1$ to $y^2 = x^3-3x^2+3x+1$ at the point $(0, 1)$, apparently by inspection without mentioning its geometric interpretation. He simply substitutes $\frac{3}{2}x + 1$ for y in $y^2 = x^3 - 3x^2 + 3x + 1$ and gets $x^3 - \frac{3}{2}x^2 = 0$. What would you substitute for y to find the tangent at $(0, 1)$ to the curve $y^2 = x^3 - 3x^2 + 5x + 1$? What is the tangent line and what is the geometric interpretation of the double root at $x = 0$?

9.3.3 Derive the formula of Hudd and Sluse by differentiating $\sum a_{ij}x^iy^j = 0$ with respect to x and showing that

$$\frac{dy}{dx} = -\frac{\sum i a_{ij}x^{i-1}y^j}{\sum j a_{ij}x^iy^{j-1}}.$$

Please hand in problems M1 – M4 on Friday, Dec.2.

M1. Please hand in the following exercises from from Stillwell's *Mathematics and its History.*

9.4.1 Use the identity $\sin x = \frac{1}{2}\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ to show that

$$\frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)} = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\cdots\cos\left(\frac{x}{2^n}\right),$$

whence

$$\frac{\sin x}{x} = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\cos\left(\frac{x}{2^3}\right)\cdots$$

9.4.2 Deduce Viète's product by substituting $x = \frac{\pi}{2}$.

9.5.3 Show that the binomial series gives

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^6 + \cdots$$

9.5.4 Use exercise 9.5.3 and

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

to derive Newton's series for $\sin^{-1} x$.

M2. Use Fermat's method of ad-equation to find the slope of the curve $f(x) = x^2 - \sqrt{x}$ at $x > 0$.

M3. Use Newton's version of Newton's method to approximate the root of $x^2 - 2 = 0$ to an accuracy of eight decimal places.

M4. On a separate piece of paper, write your Essay on Modern Mathematics proposal. It should be about a specific mathematical discovery/ theorem/ method that occurred after 1850. After the proposal is returned to you, please hand your proposal in again when you hand in your essay next week. Be sure to include in your proposal

- Working Title
- Short but specific description of what your essay is about. Don't just say you will discuss what the Henri Poincaré thought about the fundamental group. Better to describe not only why Poincaré invented the fundamental group, but say what it is and prove some things about it. Everyone in class should have a different topic.
- State an interesting fact you've discovered about your topic in your preliminary readings.
- State which style manual you'll follow. You can find a list at the Mariott website `http://campusguides.lib.utah.edu/style`
- Give two internet references. Please include the author and the URL.
- Give two book or journal references specific to your topic (other than Stillwell). One of your sources must be a primary source, quoting directly from the mathematician being discussed.

Please hand in problems N1 on Wednesday, Dec. 7.

N1. Essay on the Modern Mathematics. Write an essay about a specific mathematical discovery/ theorem/ method that occurred after 1850.

- The paper should be five pages (in some reasonable font and font size) double-spaced and printed out on paper. It should be in written in good technical English. It should be written for an audience of Math 3010 students.
- There must be some mathematics, and mathematical explanation, in your paper. Just how you incorporate some mathematical exposition will vary from subject to subject. Include displayed equations and diagrams if appropriate. You should have a good enough understanding of the mathematics you write about so that your mathematical arguments flow in a logical order and are not just an unsubstantiated and disconnected sequence of statements.
- You must draw on a bare minimum of three book and journal sources. ONE MUST BE a "primary" source, *quoting directly from the mathematician you're discussing*, or at least from sources closest to them reconstructing the original source. A "Secondary" source is a scholarly interpretation later than the original subject of study in a book or journal. You may use blogs and Wiki articles provided that you give them credit. But also track down the source cited in a Wikipedia article.
- Give credit where it is due: whenever you use another author's ideas, whether appearing in your paper as direct quotation, paraphrase, or simply influence, you must cite them (with a footnote and then include in the bibliography). Formatting these citations and bibliography entries should be unambiguous, according to your chosen style guide. (Parts of these instructions are quoted from Patrikis's assignment 2-19-16.)
- Please attach your essay proposal from last week to your paper.

The Final Exam is Wed., Dec. 14, 10:30 AM - 12:30 PM in the usual room LCB 225.