Math 3010 § 1. **First Midterm Exam** Name: Solutions
Treibergs September 25, 2024

1. *For each mathematician fill in their principal location from the list and write a short statement of their mathematical contribution.* **Locations** *(Several may be in the same location.) Alexandria, Athens, Cnidos, Croton, Elia, Miletus, Syracuse*

Mathematician	Location	Mathematical Contribution
Pythagoras 580 – 497 BC	Croton	Figurate numbers, Pythagorean triples Irrationality of $\sqrt{2}$, theory of harmony
Zeno 490 – 425 BC	Elia	Posed paradoxes, e.g., "Achilles," challenging intuition that space / time are discrete.
Plato 429 – 348 BC	Athens	Delian problems were studied in his Academy. Devised way to double the cube.
Euclid 330 – 270 BC	Alexandria	Organized all mathematics in *Elements,* a model of mathematical exposition.
Eudoxus 400 – 347 BC	Cnidos	Resolved Xeno's questions by a theory of proportion. Anticipated calculus limits.
Eratosthenes 276 – 194 BC	Alexandria	Head Librarian. Sieve of Eratosthenes. Measured circumference of Earth.
Ptolemy 100 – 170 AD	Alexandria	Wrote important treatises on astronomy, *Almagest,* and geography, *Geografike Syntaxis.*

2. (a) *Use the Babylonian method and sexigesimal arithmetic to compute the quotient* $6,40 \quad \div \quad 25$. *(Other methods receive zero credit.)*

Compute $6,40 \times \frac{1}{25}$. Find in sexigesimal

$$\frac{1}{25} = \frac{1}{5} \times \frac{1}{5} = ;12 \times ;12 = ;0,144 = ;2,24$$

since $144 = 2 \times 60 + 24$.

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     6,  40
x          ;    2,  24
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           ;  144, 960
       12 ;   80
---------------------
        2 ;  16                      960  =  16 x 60  +   0
          ;  24,    0                144  =   2 x 60  +  24
        1 ;
       12 ;  20                        80 =   1 x 60  +  20
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       15 ;  60
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       16 ;
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We can check: $6,40 = 6 \times 60 + 40 = 400$ and $\frac{400}{25} = 16$.

(b) *Using the Egyptian method of doubling, find the product* $37 \quad \times \quad 3\,\overline{8}$. *(Other methods receive zero credit.)*

$$\begin{array}{rcccl} 1 & \times & 3\,\overline{8} & = & 3\,\overline{8} \\ 2 & \times & 3\,\overline{8} & = & 6\,\overline{4} \\ 4 & \times & 3\,\overline{8} & = & 12\,\overline{2} \\ 8 & \times & 3\,\overline{8} & = & 25 \\ 16 & \times & 3\,\overline{8} & = & 50 \\ 32 & \times & 3\,\overline{8} & = & 100 \end{array}$$

Now $37 = 32 + 4 + 1$ so

$$37 \quad \times \quad 3\,\overline{8} \quad = \quad 100 + 12\,\overline{2} + 3\,\overline{8} \quad = \quad 115\,\overline{2}\,\overline{8}.$$

3. (a) *Determine whether the following statements are true or false.*

i. STATEMENT: *The Babylonians knew how to find infinitely many Pythagorean Triples.*
TRUE.

ii. STATEMENT: *The Babylonians could solve quadratic equations, such as "what are the dimensions of a rectangle whose perimeter is 31 and whose area is 55?"*
TRUE.

iii. STATEMENT: *The Greeks had many proofs of the Pythagorean Theorem.*
TRUE.

iv. STATEMENT: *The Greeks could trisect an angle using just straightedge and compass.*
FALSE.

(b) *Give a detailed explainantion of ONE of your answers (i)–(iv) above.*
You only need to supply one of these elaborations.

i. The large cuneiform tablet Plimpton 322 lists 15 huge Pythagorean triples as discussed in section 2.6. Burton argues that the Babylonians must have had general formulas to generate the entries in the table. With choices of increasing m and n, the entries follow the formulae

$$x = 2mn, \qquad y = m^2 - n^2, \qquad z = m^2 + n^2$$

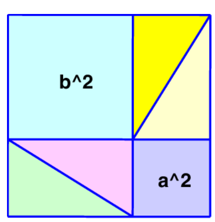
for which $x^2 + y^2 = z^2$. This scheme gives infinitely many Pythagorean triples.

ii. The Babylonians knew the quadratic formula for specific types of quadratic problems, such as the one mentioned. For this particular rectangle problem, the semiperimiter is $\frac{31}{2}$. The Babylonians may have solved the problem like this. The sides would have length $\frac{31}{4} + a$ and $\frac{31}{4} - a$ which add to the right semiperimiter. Multiplying to get the area

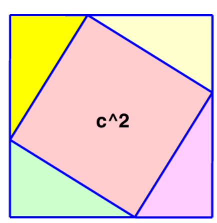
$$\begin{aligned} \left(\frac{31}{4} + a\right)\left(\frac{31}{4} - a\right) &= 55 \\ \left(\frac{31}{4}\right)^2 - a^2 &= 55 \\ a^2 &= \left(\frac{31}{4}\right)^2 - 55 = \frac{961 - 880}{16} = \frac{81}{16} \\ a &= \pm\frac{9}{4} \end{aligned}$$

so the side lengths are $\frac{31}{4} + \frac{9}{4} = 10$ and $\frac{31}{4} - \frac{9}{4} = 5.5$.

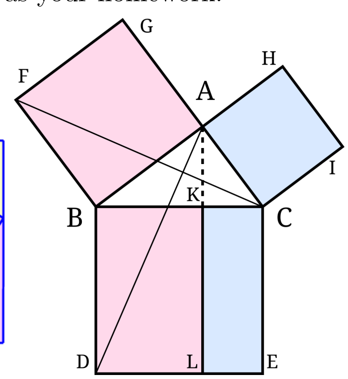
iii. The Greeks knew many proofs. We saw three of them in class, one possibly by Pythagoras and two due to Euclid, the last was your homework!

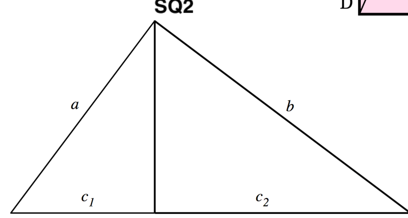


SQ1

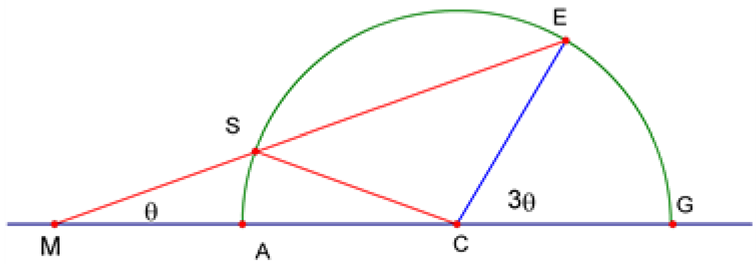


SQ2





iv. The Greeks were unable to trisect the angle using only straightedge and compass. It was proved in the nineteenth century that such constructions are impossible. However, using more complicated gadgets, the Greeks found ways to do all the Delian problems. One method of trisecting the angle is using a machine of Archimedes.



4. (a) *Use Pythagoras method to show* $\sqrt{3}$ *is irrational.*

We argue by contradiction. Assuming that $\sqrt{3}$ is rational, we may write it in lowest terms as a ratio of positive integers

$$\sqrt{3} = \frac{p}{q}$$

where p, q have no common factors other than one. Then

$$3 = \frac{p^2}{q^2}$$

or

$$p^2 = 3q^2.$$

This says 3 divides p^2. But since 3 is prime, it must divide p. Hence $p = 3k$ for some integer k. Inserting

$$(3k)^2 = 3q^2$$

yields

$$q^2 = 3k^2.$$

As before, this says that 3 divides q. We have reached a contradiction. We showed that 3 is a common factor to both p and q, contrary to our choice that p and q have no nontrivial common factors. Thus the contrary statement was false proving instead that $\sqrt{3}$ is not rational.

(b) *What is Theon of Smyrna's method to approximate* $\sqrt{2}$*, and why does it work?*
Hint: $x_{n+1} = x_n + y_n, \quad y_{n+1} = 2x_n + y_n.$
Writing $L(x, y) = y^2 - 2x^2$, Theon proved using the recursion formulas that

$$L(x_{n+1}, y_{n+1}) = -L(x_n, y_n).$$

Thus if we start with $x_1 = 2$ and $y_1 = 3$ with $L(x_1, y_1) = 1$ and follow the recursion then the alternation gives solutions of the Diophantine Equations

$$L(x_n, y_n) = (-1)^{n+1}$$

Dividing by x_n^2 gives

$$\frac{y_n^2}{x_n^2} = 2 + \frac{(-1)^{n+1}}{x_n^2}$$

which tends to 2 because $x_n \to \infty$ as $n \to \infty$. Thus

$$\frac{y_n}{x_n} \to \sqrt{2} \qquad \text{as } n \to \infty.$$

5. (a) *Find* $\gcd(165, 28)$.

$$\begin{aligned} 165 &= 5 \cdot 28 + 25 \\ 28 &= 1 \cdot 25 + 3 \\ 25 &= 8 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 + 0. \end{aligned}$$

Thus $g = \gcd(165, 28) = 1$.

(b) *Determine if there are integer solutions, and if there are, find them all*

$$165x + 28y = 1$$

g divides the right side so there are solutions. We run the Euclidean algorithm backwards

$$\begin{aligned} 1 &= 25 - 8 \cdot 3 \\ &= 25 - 8 \cdot (28 - 25) = 9 \cdot 25 - 8 \cdot 28 \\ &= 9 \cdot (165 - 5 \cdot 28) - 8 \cdot 28 = 9 \cdot 165 - 53 \cdot 28 \end{aligned}$$

One solution is of the form $x = 9$ and $y = -53$. All solutions have the form

$$x = 9 + \frac{28j}{g} = 9 + 28j, \qquad y = -53 - \frac{165j}{g} = -53 - 165j,$$

where $j \in \mathbb{Z}$ is an arbitrary integer.