Math 3010 § 1. **Second Midterm Exam** Name: Solutions ,
Treibergs March 19, 2025

1. *For each mathematician fill in their principal location from the list and write a short statement of their mathematical contribution.*

Mathematician	Location	Mathematical Contribution
Apollonius 250 – 175 BC	Alexandria	Wrote *Conics,* the culmination of Greek mathematics on conic sections.
Liu Hui 223 – 293 AD	Louyang	Wrote *Commentaries on Nine Chapters* and *Sea Island Manual* on arithmetic and geometry. Approximated π.
Brahmagupta 598 – 665 AD	Bhinmal	Quadratic formula. Euclidean algorithm. Solved linear Diophantine and Pell's equations.
Al-Khowarizmi 780 – 850 AD	Baghdad	House of Wisdom. Wrote on arithmetic, astronomy and ... *al-Jabr wal muqabala* on quadratic equations.
al-Khayyami 1048 – 1131 AD	Baghdad	His treatise advanced algebra. Commentaries on Euclid. Geometric solution of cubic equations.
Qin Jiushao 1202 - 1261 AD	Hangzhou	*Mathematical Teatise in Nine Sections.* High order eqns. Binomial theorem. Ta-Yen rule for congruences.
Francoise Viete 1540 – 1603 AD	Tours	Advanced algebraic notation. Found trig. expansions and used trig. identities to solve cubic equations.

Locations (Several may be in the same location.)

Alexandria, Athens, Baghdad, Bhinmal, Bologna Hangzhou, Luoyang, Tours, Ujjian

2. (a) *Find positive integers x and y such that* $23\,x^2 + 1 = y^2$.
We are close if we take $(x, y, k) = (1, 5, 2)$,

$$32x^2 + k = y^2$$
$$23 \cdot 1^2 + 2 = 5^2.$$

Applying Brahmagupta's thunderbolt for two sets of solutions (x_1, y_1, k_1) and (x_2, y_2, k_2) with $D = 23$,

$$(x_1, y_1, k_1) \oplus (x_2, y_2, k_2) = (x_1y_2 + x_2y_1, Dx_1x_2 + y_1y_2, k_1k_2)$$
$$(1, 5, 2) \oplus (1, 5, 2) = 1 \cdot 5 + 5 \cdot 1, 23 \cdot 1^2 + 5^2, 2 \cdot 2) = (10, 48, 4)$$

which gives the solution

$$23 \cdot 10^2 + 4 = 48^2.$$

Dividing by four yields the sought-for solution

$$23 \cdot 5^2 + 1 = 24^2,$$

i.e., $x = 5$ and $y = 24$.

(b) *Find all integers N satisfying the simultaneous congruences*

$$\begin{aligned} N &\equiv 3 \mod 5 \\ N &\equiv 5 \mod 6 \\ N &\equiv 2 \mod 7 \end{aligned}$$

One notes that 5, 6 and 7 are pairwise relatively prime, so that simultaneous solution exists.$6 \cdot 6 \cdot 7 = 210$. The first method is to solve congruences pairwise. Starting from the first two, there are integers x and y so that

$$3 + 5x = N = 5 + 6y$$

which is equivalent to

$$5x - 6y = 2.$$

One sees by inspection that $x = -2$, $y = -2$ is a solution. Thus all solutions have the form

$$x = -2 + 6j, \qquad y = -2 + 5j$$

where j is an arbitrary integer. Using the the last two congruences, one finds there are integers y, j z

$$5 + 6(-2 + 5j) = 5 + 6y = N = 2 + 7z,$$

which simplifies to

$$30j - 7z = 9.$$

If one takes $j = -2$ and $z = -9$ we have $30(-2) - 7(-9) = 3$ so that tripling gives a solution $30(-6) - 7(-27) = 9$. Thus the general solution of the second congruence is

$$y = -6 + 7k, \qquad z = -27 + 30k$$

where k is an arbitrary integer. This shows that

$$N = 2 + 7z = 2 + 7(-27 + 30k) = -187 + 210k.$$

If we use $k = k' + 1$ instead, we get a smaller basic solution

$$N = 23 + 210k',$$

where k' is an arbitrary integer.

The second method is Qin Jiushao's Ta-Yen Rule. We see that

$$\begin{aligned} N_1 &= 6 \cdot 7 = 42 \equiv 2 \mod 5, & N_1 &\equiv 0 \mod 6, & N_1 &\equiv 0 \mod 7 \\ N_2 &= 5 \cdot 7 = 35 \equiv 5 \mod 6, & N_2 &\equiv 0 \mod 5, & N_2 &\equiv 0 \mod 7 \\ N_3 &= 5 \cdot 6 = 30 \equiv 2 \mod 7, & N_3 &\equiv 0 \mod 5, & N_3 &\equiv 0 \mod 6 \end{aligned}$$

N_2 and N_3 are as desired. Also

$$4 \cdot N_1 = 4 \cdot 42 = 168 \equiv 4 \cdot 2 \mod 5 \equiv 3 \mod 5$$

Thus the solution of the simultaneous congruences is

$$N = 4 \cdot N_1 + N_2 + N_3 = 168 + 35 + 30 = 233$$

modulo 210, which is the same as before.

3. (a) *Determine whether the following statements are true or false.*

   i. STATEMENT: *Napier's logarithms reduced the computation of a product of two positive numbers x and y to the formula* $\mathrm{Nlog}(xy) = \mathrm{Nlog}(x) + \mathrm{Nlog}(y)$.
   FALSE.

   ii. STATEMENT. *Ibn Munim considered the number of words with permutations that could be formed out of the 28 letters of the Arabic alphabet. How many words of ten letters are there, each word having two nonrepeated letters, one letter repeated twice and two letters repeated three times, e.g., $ABCCDDDEEE$?*
   TRUE.

   iii. STATEMENT. *Regiomontanus developed the law of sines in his book On Triangles, and answered: if the angles at two vertices of a triangle are $55°$ and $66°$ and the length of the side between them is 10, what are the lengths of the other two sides?*
   TRUE.

   iv. STATEMENT. *The Chinese gave an algorithm to find square root, say of of $N = 46,656$, in* Nine Chapters on the Mathematical Art.
   TRUE.

(b) *Give a detailed explanation of ONE of your answers (i)–(iv) above.*

   i. Napier did develop logarithms and produced tables of logarithms of numbers and logarithms of sines of angles. However the formula is incorrect for Napierian Logarithms. Recall that Napier used $r = 10^7$ and $b = 1 - 10^{-7}$ as base for his logarithms. Thus $\ell = \mathrm{Nlog}(x)$ means $x = r\, b^\ell$. If for instance, if $\ell = \mathrm{Nlog}(x)$ and $m = \mathrm{Nlog}(y)$ then
$$x = r\, b^\ell, \qquad y = r\, b^m$$
   then the correct formulas are
$$10^{-7}xy = r^{-1}\, r\, b^\ell\, r\, b^m = rb^{\ell+m}$$
   so
$$\mathrm{Nlog}(10^{-7}xy) = \ell + m = \mathrm{Nlog}\, x + \mathrm{Nlog}\, y$$
   or using $p = \mathrm{Nlog}\, 1$ which means $1 = rb^p$ or $r = b^{-p}$ so
$$xy = r\, b^\ell\, r\, b^m = r^2 b^{\ell+m} = r\, b^{\ell+m-p}$$
   so that
$$\mathrm{Nlog}(xy) = \ell + m - p = \mathrm{Nlog}\, x + \mathrm{Nlog}\, y - \mathrm{Nlog}\, 1.$$

   ii. Ibn Munim counted numbers of such words. There are 28 choices for the first triply repeated letter and $\binom{10}{3}$ subsets of size three in ten letters where to put the three letters. Then there are 27 remaining choices for the second triply repeated letter and $\binom{7}{3}$ remaining subsets of size three in the remaining seven unassigned places. This procedure double counts the triply occurring letters since the letters may be chosen in either order, so we must divide by two. There are 26 choices for the doubly occurring letter which is placed in any one of the $\binom{4}{2}$ subsets of size two left. Then there are 25 choices for the leftmost remaining letter and 24 choices for the last place. All told, there are
$$N = 28 \cdot \binom{10}{3} \cdot 27 \cdot \binom{7}{3} \cdot \frac{1}{2} \cdot 26 \cdot \binom{4}{2} \cdot 25 \cdot 24 = 148,599,360,000$$
   possible such words.

iii. Regiomontanus did write a trigonometry treatise *On Triangles* featuring solving triangles in the plane and in the sphere. If a, b, c are the sides of a planar triangle and α, β, γ are the corresponding opposite angles, then the law of sines states

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

We are given $\alpha = 55°$, $\beta = 66°$ and $c = 10$. The remaining angle is thus

$$\gamma = 180° - \alpha - \beta = 180° - 55° - 66° = 59°$$

The sine law then tells us that the remaining sides are given by

$$a = \frac{c\sin\alpha}{\sin\gamma} = \frac{10\sin 55°}{\sin 59°}, \qquad b = \frac{c\sin\beta}{\sin\gamma} = \frac{10\sin 66°}{\sin 59°}.$$

iv. A method to find square roots is described in Chapter IV *Shao guang* (*What Width?)* of the text *Nine Chapters on the Mathematical Art.* $N = 46,656 < 1000^2$ so we look for a square root of the form $x = 100a + 10b + c$ where a, b and c are integers from 0 to 9. Observe that

$$200^2 = 40,000 < 46,656 < 90,000 = 300^2$$

so $a = 2$ and $x = 200 + 10b + c$. Try to find the largest b so that

$$(200 + 10b)^2 = 40,000 + 4000b + 100b^2 \leq 46,656$$

or

$$4000b + 100b^2 \leq 46,656 - 40,000 = 6656.$$

We have $4000 \cdot 1 + 100 \cdot 1^2 = 4100$ and $4000 \cdot 2 + 100 \cdot 2^2 = 8400$ so $b = 1$ because $b = 2$ is too large. Thus

$$46,656 - 210^2 = (46,656 - 40,00) - (4000 \cdot 1 + 100 \cdot 1^2) = 6656 - 4100 = 2546.$$

Finally

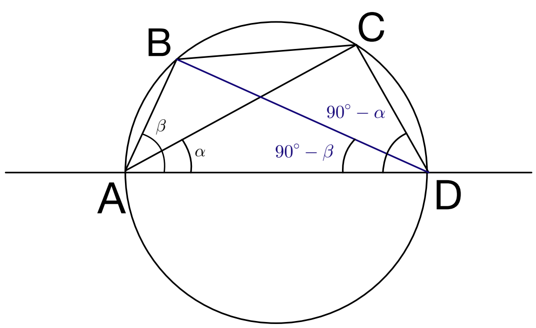
$$(210 + c)^2 = 210^2 + 420c + c^2 \leq 46,656$$

implies

$$420c + c^2 \leq 2546.$$

Since $420 \cdot 6 + 6^2 = 2520 + 36 = 2556$ we have $c = 6$ so the square root of $N = 46,656$ is $x = 216$.

4. (a) *Suppose the quadrilateral $ABCD$ is inscribed in a circle of radius R and that AD is the diameter. Let $\alpha = \angle CAD$ and $\beta = \angle BAD$. Show that the chord CD has length $2R\sin\alpha$. Using Ptolemy's Theorem, show* $\sin(\beta-\alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha$, *Hints:* $\cos\alpha = \sin(90^\circ - \alpha)$



Since A and D are endpoints of a diameter, the angle $\angle ACD = 90^\circ$ is a right angle. Thus the length $CD = AD\sin\alpha = 2R\sin\alpha$. It is a property of circles that an arc such as CD subtends the same angle at all points of the circle, thus, for example $\angle CBD = \alpha$. The arc BC viewed from A has angle $\beta - \alpha$ so the length $BC = 2R\sin(\beta-\alpha)$. Also, the angle $\angle CDA$ is the complementary angle to α so $\angle CDA = 90^\circ - \alpha$. Similarly $\angle BDA = 90^\circ - \beta$.

Ptolemy's Theorem says that for a quadrilateral whose vertices are on a circle, the sums of products of lengths of opposite sides equals the product of the diagonals

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Substitute the lengths by their trigonometric equivalents

$$2R\sin(90^\circ - \beta) \cdot 2R\sin\alpha + 2R\sin(\beta-\alpha) \cdot 2R = 2R\sin(90^\circ - \alpha) \cdot 2R\sin\beta$$

Dividing by $4R^2$ and using the complementary angle identity $\cos\alpha = \sin(90^\circ - \alpha)$ yields

$$\cos\beta\ \sin\alpha + \sin(\beta - \alpha) = \cos\alpha\ \sin\beta,$$

the desired subtraction formula for sines.

(b) *Solve using Cardano's method:* $x^3 = 9x + 12$.

Substituting $x = u + v$ we have

$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = u^3 + v^3 + 3uvx = 9x + 12$$

Thus equating the constant and x terms

$$3uv = 9$$
$$u^3 + v^3 = 12.$$

By the first equation,

$$v = \frac{3}{u}$$

and by the second

$$u^3 + \frac{27}{u^3} = 12.$$

Thus

$$u^6 - 12u^3 + 27 = 0.$$

This is quadratic in u^3. By the quadratic formulas

$$u^3, v^3 = \frac{12 \pm \sqrt{12^2 - 4 \cdot 27}}{2} = 6 \pm \sqrt{6^2 - 27} = 6 \pm \sqrt{9} = 6 \pm 3 = 3,\ 9.$$

It follows that a solution of the cubic equation is

$$x = u + v = 3^{1/3} + 9^{1/3}.$$

Check: $x^3 = u^3 + v^3 + 3uv(u+v) = 3 + 9 + 3 \cdot 3x = 12 + 9x$.

5. (a) *Describe the models of the solar system supported by the following mathematicians. How are the models distinct from the others?*

Mathematician / Model	Comparison to the others?
Claudius Ptolemy, 100-178 AD, Alexandria Geocentric planetary system with planets and sun traveling on epicycles of an eccentric circle.	This model was originally proposed by Apollonius who lacked the trigonometry to compute its dimensions, Hipparchus carried out the astronomical observations and calculated the parameters in the model. Ptolemy collected all the observed data and planar and spherical trigonometry and presented Hipparchus calculations for the motions of the sun, moon and planets in his treatise, *Almagest.* Ptolemy was one of the first scientists to use his mathematical model and make the difficult calculations required to predict the future position for each heavenly body. This model is computationally difficult.
Nicolaus Copernicus, 1473 - 1543 Warmia, Poland Heliocentric model. The universe consists of nested spheres about the sun containing the planets. The earth and planets move on epicycles about circles that revolve around the sun.	Although the notion that the universe consisted of a system of nested spheres centered on the earth was still accepted as the nature of the universe, by the fifteenth century, Islamic and Jewish astronomers and Regiomontanus had noted discrepancies between Ptolemy's predictions and their own observations, and made adjustments to Ptolemy's details. By the Renaissance, it was recognized that the true solar year was $11\frac{1}{4}$ minutes less that the $365\frac{1}{4}$ days used to set the Julian calendar. Copernicus recognized that the various pieces of the Ptolemaic model simply couldn't be patched together to make all planets move as observed. Copernicus model is explained in his treatise *On the Revolutions of Celestial Spheres* of 1543 which collected the trigonometric background and computations as in the Almagst. The observed retrograde motion of a planet outside the orbit of the earth relative to the stars can be explained be the relative motions of the earth and the planet. To agree with observation, Copernicus also made planets move on epicycles of circles centered on the sun. The resulting theory is about as complicated as the Ptolemaic one.

Mathematician / Model	Comparison to the others?
Galileo Galilei, 1564 - 1642 Padua Heliocentric model of Copernicus.	Galileo developed the physics of moving bodies. He saw the moons of Jupiter and phases of Venus through his telescope which enhanced his belief in the Copernican model. His treatise in Italian *Two Chief Systems* of 1632 was a dialog making a spirited defense of the Copernican systems and *Two New Sciences* of 1638 a dialog on the mathematics of dynamics. In 1616, Galileo was brought before the Inquisition. Pope Paul V instructed Cardinal Bellarmine to order him to abandon heliocentrism and henceforth not to hold, teach, or defend it in any way whatever, either orally or in writing. The decree of the Congregation of the Index banned Copernicus's De Revolutionibus and other heliocentric works until correction.
Johannes Kepler, 1571 - 1639, Prague Heliocentric model. The planets travel on ellipses whose focus is the sun.	Using the better astronomical data of Tycho Brahe, Kepler concluded that an elliptical orbit of mars fits the observations much better than Copernican off-center epicycles. Kepler confirmed that the same was true of the other planets. In additions, Kepler noticed that the planets follow three laws: planets orbit on ellipses with sun at the focus, planets sweep out the same area in the same time, and the period is proportional to 3/2 power of the major axis of the orbit. Kepler's laws were entirely empirical (based on observation.) In his *New Astronomy* of 1609, Kepler detailed his eight years of computations. Kepler tried to understand the celestial motion in terms of gravitational forces. Theoretical models have to be consistent with observations.

(b) *If $F_0 = 1$, $F_1 = 1$, what are the next five Fibonacci Numbers? Show that for all n*

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

The recursion is $F_{n+2} = F_{n+1} + F_n$. Thus the next five Fibonacci Numbers are

$$F_0 = 1, \quad F_1 = 1, \quad F_2 = 2, \quad F_3 = 3, \quad F_4 = 5, \quad F_5 = 8, \quad F_6 = 13, \ \ldots$$

Theorem. *For all $n \ge 0$ we have*

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1. \tag{1}$$

Proof. We argue by induction. For the base case $n = 0$ we have

$$\text{LHS.} = F_0 = 1, \qquad \text{RHS.} = F_2 - 1 = 2 - 1 = 1$$

which are equal. Thus (1) holds for $n = 0$.

Assume the induction hypothesis: that (1) holds for some n. To show that it holds for $n+1$, we see using the induction hypothesis and the recursion formula,

$$\begin{aligned} F_0 + F_1 + F_2 + \cdots + F_n + F_{n+1} &= \big(F_0 + F_1 + F_2 + \cdots + F_n\big) + F_{n+1} \\ &= (F_{n+2} - 1) + F_{n+1} \\ &= (F_{n+2} + F_{n+1}) - 1 \\ &= F_{n+3} - 1 \end{aligned}$$

which is (1) for $n+1$. Thus we have shown that the induction case holds as well. By mathematical induction (1) holds for all $n \geq 0$. $\square$