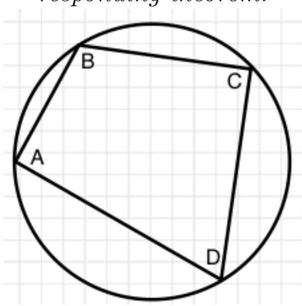
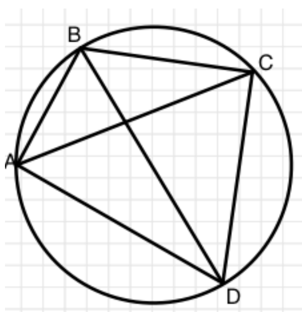
Math 3010 § 1. **Second Midterm Exam** Name: Solutions
Treibergs Mar. 18, 2026

1. *For each diagram, identify the mathematician associated with the figure and state the corresponding theorem.*

a. 

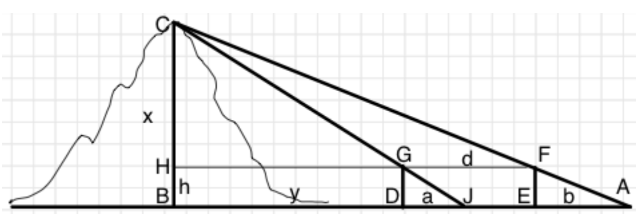
Brahmagupta 575-645 Bhinmal
Formula for the area of a circular quadrilateral in terms of the side lengths.

$$s = \frac{AB + BC + CD + DA}{2}$$
$$A = \sqrt{(s - AB)(s - BC)(s - CD)(s - DA)}.$$

b. 

Ptolemy 100-178 Alexandria
Ptolemy's Formula for products of sides of a circular quadrilateral

$$AC \cdot BD = AB \cdot CD + BC \cdot DA.$$

c. 

Liu Hui 223-295 Cao Wei. From *Sea Island Mathematical Manual* appendix to his *Commentaries on Nine Chapters.*
Calculation of height and distance of mountain top from sightings from two poles of height h, d apart.

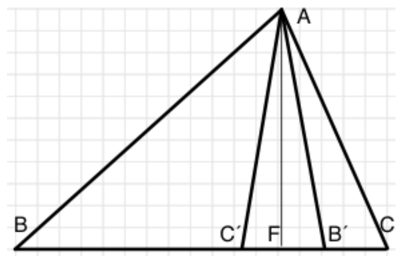
The height of the mountain is $BC = h + x$. Its distance from D is $BD = y$. The trinagles ΔCHG and ΔGDJ are similar as are triangles ΔCGF and ΔCJA. This leads to the equations

$$\frac{x}{y} = \frac{h}{a}, \qquad \frac{x}{d} = \frac{x + h}{d + b - a}.$$

Hence the height and distance are

$$x + h = \frac{hd}{b - a} + h, \qquad y = \frac{ad}{b - a}.$$

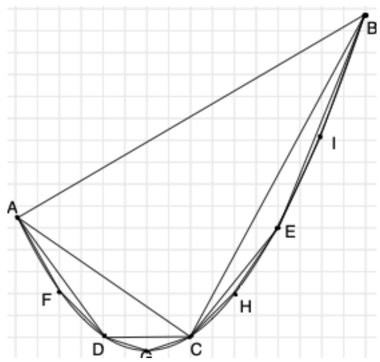
d.



Thâbit ibn Qurra 836-901 Baghdad
Generalization of Pythagorean Theorem

$$AB^2 + AC^2 = BC \cdot (BB' + CC').$$

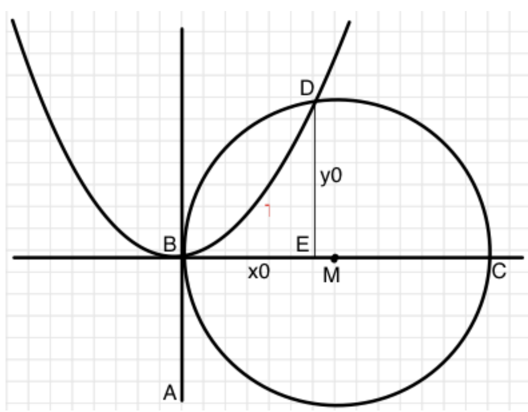
e.



Archimedes 278 - 212 BC Syracuse
Quadrature of the parabola.

$$\text{Area(Parabolic Sector } AFD \ldots B)$$
$$= \frac{4}{3} \text{Area(Triangle } \Delta ACB)$$

e.



Omar Khayyam 1084-1123 Baghdad
Geometric solution of cubic equation.

$$x^3 + b^2 x = b^2 c.$$

The line segment AB has length b. The parabola constructed with directrix distance b has the equation $x^2 = by$. The circle with diameter $BC = c$ has equation

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}.$$

The coordinate $BE = x_0$ of the intersection point $D = (x_0, y_0)$ satisfis the cubic equation.

2. (a) *Find positive integers x and y such that* $\qquad x^2 = 39y^2 + 1.$

We are close if we take $(x, y, k) = (6, 1, -3)$,

$$\begin{aligned} x^2 &= 39y^2 + k \\ 6^2 &= 39 \cdot 1^2 - 3. \end{aligned}$$

Applying Brahmagupta's thunderbolt for two sets of solutions (x_1, y_1, k_1) and (x_2, y_2, k_2) with $D = 39$,

$$\begin{aligned} (x_1, y_1, k_1) \oplus (x_2, y_2, k_2) &= (x_1x_2 + Dy_1y_2, x_1y_2 + x_2y_1, k_1k_2) \\ (6, 1, -3) \oplus (6, 1, -3) &= (6^2 + 39 \cdot 1^2, 1 \cdot 5 + 5 \cdot 1, (-3) \cdot (-3)) = (75, 12, 9) \end{aligned}$$

which gives the solution

$$75^2 = 39 \cdot 12^2 + 9.$$

Dividing by nine yields the sought-for solution

$$25^2 = 39 \cdot 4^2 + 1,$$

i.e., $x = 25$ and $y = 4$.

(b) *Find all integers N satisfying the simultaneous congruences*

$$\begin{aligned} N &\equiv 1 \mod 5 \\ N &\equiv 3 \mod 7 \\ N &\equiv 2 \mod 17 \end{aligned}$$

One notes that 5, 7 and 17 are pairwise relatively prime, so that simultaneous solution N exists and is given modulo $5 \cdot 7 \cdot 17 = 595$. The first method is to solve congruences pairwise. Starting from the first two, there are integers x and y so that

$$1 + 5x = N = 3 + 7y$$

which is equivalent to

$$5x - 7y = 2.$$

One sees by inspection that $x = -1$, $y = -1$ is a solution. Thus all solutions have the form

$$x = -1 + 7j, \qquad y = -1 + 5j$$

where j is an arbitrary integer. Using the the last two congruences, there are integers $y = -1 + 5j$ and z such that

$$3 + 7(-1 + 5j) = 3 + 7y = N = 2 + 17z,$$

which simplifies to

$$35j - 17z = 6.$$

If one takes $j = 1$ and $z = 2$ we have $35(1) - 17(2) = 1$ so that multiplying by 6 gives a solution $35(6) - 17(12) = 6$. Thus the general solution of the second congruence is

$$j = 6 + 17k, \qquad z = 12 + 35k$$

where k is an arbitrary integer. This shows that

$$N = 2 + 17z = 2 + 17(12 + 35k) = 206 + 595k.$$

This is a solution since $206 = 41 \cdot 5 + 1 = 29 \cdot 7 + 3 = 12 \cdot 17 + 2$.
The second method is Qin Jiushao's Ta-Yen Rule. We see that

$$\begin{aligned} N_1 = 7 \cdot 17 = 119 &\equiv 4 \mod 5, & N_1 &\equiv 0 \mod 7, & N_1 &\equiv 0 \mod 17 \\ N_2 = 5 \cdot 17 = 85 &\equiv 1 \mod 7, & N_2 &\equiv 0 \mod 5, & N_2 &\equiv 0 \mod 17 \\ N_3 = 5 \cdot 7 = 35 &\equiv 1 \mod 17, & N_3 &\equiv 0 \mod 5, & N_3 &\equiv 0 \mod 7 \end{aligned}$$

N_2 and N_3 are as desired. Also

$$-N_1 = -119 \equiv 1 \mod 5$$

Thus the solution of the simultaneous congruences is

$$N = 1 \cdot (-N_1) + 3 \cdot N_2 + 2 \cdot N_3 = -119 + 255 + 70 = 206$$

modulo 595, which is the same as before.

3. (a) *Determine whether the following statements are true or false.*

   i. STATEMENT: *For x, y positive, the Napierian logarithms satisfy* $\text{Nap.}\log(xy) = \text{Nap.}\log(x) + \text{Nap.}\log(y)$.
   FALSE.

   ii. STATEMENT: *Archimedes showed* $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
   TRUE.

   iii. STATEMENT: *Oresme could sum the series* $\sum_{n=1}^{\infty} \frac{n}{2^n}$.
   TRUE.

   iv. STATEMENT: *The Chinese had a method to determine the square root of* $N = 51,529$.
   TRUE.

   (b) *Give a detailed explanation of ONE of your answers i.– iv. above.*

   i. Napier developed logarithms and produced tables of logarithms of numbers and logarithms of sines of angles. Recall that Napier used $r = 10^7$ and $b = 1 - 10^{-7}$ as base for his logarithms. Thus $\ell = \text{Nap.}\log(x)$ means $x = r\, b^{\ell}$. If for instance, if $\ell = \text{Nap.}\log(x)$ and $m = \text{Nap.}\log(y)$ then

   $$x = r\, b^{\ell}, \qquad y = r\, b^{m}.$$

   Using $p = \text{Nap.}\log 1$ which means $1 = rb^p$ or $r = b^{-p}$ so

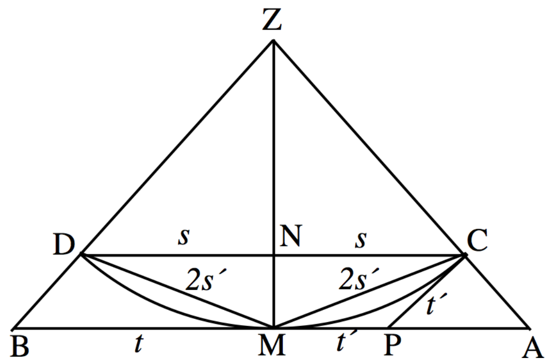
   $$xy = r\, b^{\ell}\, r\, b^{m} = r^2 b^{\ell+m} = r\, b^{\ell+m-p}$$

   so that

   $$\text{Nap.}\log(xy) = \ell + m - p = \text{Nap.}\log x + \text{Nap.}\log y - \text{Nap.}\log 1$$

   and not the formula in the question

ii. Archimedes found upper and lower bounds on π by computing the lengths of regular polygons that inscribe and circumscribe a circle of radius r. Let A_k and B_k denote the perimeter of the circumscribed and inscribed $3 \cdot 2^k$-gon. He found recursion formulae for the perimeters and calculated up to $k = 5$ which are 96-gons. By carefully tracking errors, he found $3\frac{10}{71} < B_5 < \pi < A_5 < 3\frac{1}{7}$.



Let us recall Archimedes the recursion formulae. In the diagram Z is the origin and $ZDMC$ is the sector corresponding to the $n = 3 \cdot 2^k$-gon. Let $DC = 2s$ and $BA = 2t$ be the lengths of one side of the inscribed and circumscribed n-gon. By dividing this sector along ZM we get two sectors of the $2n$-gon whose side lengths are $2s'$ and $2t'$, resp. Then the total lengths of the circumscribed and inscribed n and $2n$ gons are

$$A_k = 2nt, \qquad B_k = 2ns, \qquad A_{k+1} = 4nt', \qquad B_{k+1} = 4ns'. \tag{1}$$

Let M be the midpoint of AB, N the midpoint of CD and P be the point of intersection of MA and the tangent to the circle passing through C. Thus $MP = PC = t'$ is half the side of the circumscribing $2n$-gon and $MC = MD = 2s'$ the side of the inscribed $2n$-gon.
Since ΔACP and ΔAMZ are similar triangles, we have

$$\frac{t'}{t - t'} = \frac{PC}{PA} = \frac{ZM}{ZA}.$$

Also ΔZNC and ΔZMA are similar triangles, we have using $ZM = ZC$

$$\frac{s}{t} = \frac{NC}{MA} = \frac{ZC}{ZA} = \frac{ZM}{ZA}$$

Since the right sides are equal, we have $t'/(t - t') = s/t$ or

$$t' = \frac{ts}{t + s}.$$

Since the isosceles triangles ΔCMD and ΔCPM are similar,

$$\frac{2s'}{2s} = \frac{t'}{2s'}$$

or

$$2(s')^2 = st'.$$

Subsituting (1) we ontain the recursion formulae of Archimedes

$$A_{k+1} = \frac{2A_k B_k}{A_k + B_k}, \qquad B_{k+1} = \sqrt{B_k A_{k+1}}.$$

The initial values correspond to the hexagon $B_1 = 4\sqrt{3}r$ and $A_1 = 6r$. Archimedes calculations were laborious, since he had to do square roots by hand. Just for fun, here are ©R computations for the circle of radius $r = 1/2$.

```
         k          a        b
      [1,] 3.464102 3.000000
      [2,] 3.215390 3.105829
      [3,] 3.159660 3.132629
      [4,] 3.146086 3.139350
      [5,] 3.142715 3.141032
      [6,] 3.141873 3.141452
      [7,] 3.141663 3.141558
      [8,] 3.141610 3.141584
      [9,] 3.141597 3.141590
     [10,] 3.141594 3.141592
     [11,] 3.141593 3.141593
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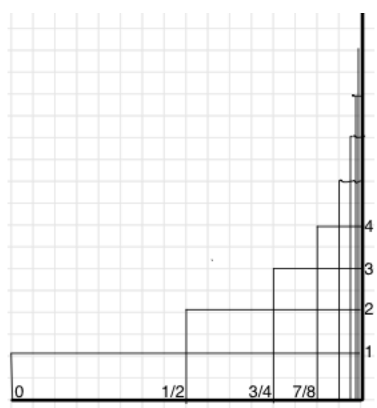
[From H. Dörrie, *100 Great Problems of Elementary Mathematics,* Dover, 1965, pp. 184-188.]

iii. Nicole Oresme (1320-1382) had a trick for summing $S = \sum_{n=1}^{\infty} \frac{n}{2^n}$. S is the total area of the figure, which has height 1 for $0 \le x < \frac{1}{2}$, height 2 for $1/2 \le x < 3/4$, 3 for $3/4 \le x < 7/8$ and so on. Each column has height k and width 2^{-k}. Adding up the columns gives the total area S. But the area can also be added a row at a time. The first row $0 \le y \le 1$ for $0 \le x \le 1$ has width 1. The second row $1 < y \le 2$ for $1/2 \le x \le 1$ has width 1/2. The third row $2 < y \le 3$ for $3/4 \le x \le 1$ has width 1/4 and so on. Thus when added row wise

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 2$$

which is a geometric sum.



iv. A method to find square roots is described in Chapter IV *Shao Guang* (*What Width?)* of the text *Nine Chapters on the Mathematical Art.* To approximate the square root of $N = 51,529$ find the digits one at a time, by solving the linear part of the correction. We observe that

$$100^2 = 10,000 < N = 51,529 < 1,000^2 = 1,000,000$$

so that we can seek a square root of the form $x = 100a + 10b + c$ where a, b, c are integers from 0 to 9. We observe that

$$200^2 = 40,000 < N = 51,529 < 300^2 = 90,000$$

so that $a = 2$. We seek the largest b such that

$$N = 51,529 \geq (200 + 10b)^2 = 40,000 + 4,000b + 100b^2$$

Subtracting, the main term is

$$N - 200^2 = 51,529 - 40,000 = 11,529 \geq 4,000b$$

$b = 3$ is too large. Trying $b = 2$ we have $4,000b + 100b^2 = 8,400 < N - 200^2$ and $N - 220^2 = 11,529 - 8,400 = 3,129$. Next we seek c such that

$$N = 51,529 \geq (220 + c)^2 = 220^2 + 440c + c^2$$

Now we seek the largest c so that

$$N - 220^2 = 3,129 \geq 440c + c^2$$

With $7 = 3$ we obtain $440c + c^2 = 3129$. Thus $x = 227$ and $x^2 = 51,529$.

4. (a) *Explain what is the Fibonacci Sequence:* $F_1, F_2, F_3, \ldots$. *Show that for each* n *the Fibonacci numbers satisfy*

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}. \tag{2}$$

The Fibonacci Sequence is defined recursively $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 3$. The first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. The problem is described in Leonardo of Pisa's 1201 book *Liber Abaci* concerning some fecund pairs of rabbits.

The first argument uses induction. For the base case, check that the equation (2) holds for $n = 1$. We have

$$\text{LHS} = \sum_{l=1}^{1} F_k^2 = f_1^2 = 1^2, \qquad \text{RHS} = F_1 F_2 = 1 \cdot 1.$$

For the induction case, assume that (2) folds for $n \geq 2$ to show it also holds for $n + 1$. Using the induction hypothesis (2) and the recursion $F_{n+1} = F_n + F_{n-1}$,

$$\begin{aligned}\sum_{k=1}^{n+1} F_k^2 &= \left(\sum_{k=1}^{n} F_k^2\right) + F_{n+1}^2 \\ &= F_n F_{n+1} + F_{n+1}^2 \\ &= F_{n+1}(F_n + F_{n+1}) \\ &= F_{n+1} F_{n+2}\end{aligned}$$

which is (2) for $n + 2$. Since both the base and induction cases hold, by mathematical induction (2) holds for all n.

The second argument uses telescoping sums. The recursion may be written $F_k = F_{k+1} - F_{k-1}$. This even works for $k = 1$ if we interpret $F_0 = 0$. Thus

$$\begin{aligned}\sum_{k=1}^{n} F_k^2 &= \sum_{k=1}^{n} F_k(F_{k+1} - F_{k-1}) = \sum_{k=1}^{n} \{F_k F_{k+1} - F_{k-1} F_k\} \\ = \{F_1 F_2 - F_0 F_1\} &+ \{F_2 F_3 - F_1 F_2\} + \{F_3 F_4 - F_2 F_3\} + \cdots + \{F_n F_{n+1} - F_{n-1} F_n\} \\ &= F_n F_{n+1} - F_0 F_1 = F_n F_{n+1}.\end{aligned}$$

(b) *Find a root of the cubic equation $x^3+18x = 4$. Following Cardano, substitute $x = a-b$ and solve for a and b. Don't just plug into Cardano's formula.*

Substituting $x = a - b$ we have

$$(a-b)^3+18(a-b) = a^3-3a^2b+3ab^2-b^3+18(a-b) = a^3-3ab(a-b)-b^3+18(a-b) = 4$$

Thus equating the constant and $a - b$ terms

$$\begin{aligned} a^3 - b^3 &= 4 \\ 3ab &= 18 \end{aligned}$$

Squaring the first equation and cubing the second $ab = 6$

$$\begin{aligned} a^6 - 2a^3b^3 + b^6 &= 16 \\ a^3b^3 &= 216 \end{aligned}$$

Adding four times the second to the first

$$(a^3 + b^3)^2 = a^6 + 2a^3b^3 + b^6 = 16 + 4 \cdot 216 = 880 = 16 \cdot 55.$$

Thus this and the first equation yield

$$\begin{aligned} a^3 - b^3 &= 4 \\ a^3 + b^3 &= 4\sqrt{55} \end{aligned}$$

Solving we find

$$\begin{aligned} a^3 &= 2 + 2\sqrt{55} \\ b^3 &= -2 + 2\sqrt{55} \end{aligned}$$

Thus the solution to the cubic is

$$x = a - b = \sqrt[3]{2 + 2\sqrt{55}} - \sqrt[3]{-2 + 2\sqrt{55}}.$$

This is the answer. It turns out $x = 0.2216175$ which checks since $x^3 + 18x = 4$.

5. (a) *What is a Diophantine Equation? Solve the following problem from* Arithmetica *of Diophantus: find four numbers such that when any three of them are added together, their sum is one of the four numbers 20, 22, 24 and 27.*

An equation or a system of equations are called *Diophantine* if it is a polynomial system with integer or rational coefficients and the solution is required to be integer or rational. Often the system is underdetermined, there are more variables than equations with the condition that solutions be rational making up for the missing equation. Examples of Diophantine equations are the linear Diophantine equation $ax + by = c$, the equation for Pythagorean Triples $x^2 + y^2 = z^2$ and Pell's Equation $x^2 = Dy^2 + 1$.

To find four numbers, following the text, we adapt Diophantus's trick: to solve for the sum $s = x_1 + x_2 + x_3 + x_4$. Thus the four numbers are $s-20$, $s-22$, $s-24$ and $s-27$ whose sum is

$$s = (s - 20) + (s - 22) + (s - 24) + (s - 27) = 4s - 93.$$

Solving, $3s = 93$ or $s = 31$ and the four numbers are 11, 9, 7 and 4.

Of course you can formulate the problem as a four by four system of linear equations for the four numbers, but that takes longer.

(b) *Describe the models of the solar system advocated by Ptolemy, al-Tûsî and Copernicus. What are the strengths and weaknesses of each model?*

Mathematician / Model	Comparison to the others?
Claudius Ptolemy, 100-178 AD, Alexandria Geocentric planetary system with planets traveling on epicycles of an eccentric circle.	This model was originally proposed by Apollonius who lacked the trigonometry to compute its dimensions, Hipparchus carried out the astronomical observations and calculated the parameters in the model. Ptolemy collected all the observed data and planar and spherical trigonometry and presented Hipparchus calculations for the motions of the sun, moon and planets in his treatise, *Almagest.* Ptolemy was one of the first scientists to use his mathematical model and make the difficult calculations required to predict the future position for each heavenly body. This model is computationally difficult.
Nasîr al-Din al-Tûsî, 1201-1274, Maragha, Iran Geocentric planetary system with planets and traveling on epicycles using mathematics that improved Ptolemy's description.	at-Tûsî invited the most renowned astronomers, some from as far away as Spain and China. They calculated new astronomical tables of planetary movements in 1271. The Maragha astronomers developed a non-Ptolemaic theory of planetary motion. al-Tûsî invented a geometric technique called the Tusi Couple to replace Ptolemy's problematic equant. al-Tûsî criticized Ptolemy's use of observational evidence to show that the Earth was at rest, noting that such proofs were not decisive. Although it doesn't mean that he was a supporter of mobility of the earth. He maintained that the earth's immobility could be demonstrated only by physical principles. al-Tûsî's criticisms of Ptolemy were similar to the arguments later used by Copernicus in 1543 to defend the Earth's rotation.
Nicolaus Copernicus, 1473 - 1543 Warmia, Poland Heliocentric model. The universe consists of nested spheres about the sun containing the planets. The earth and planets move on epicycles about circles that revolve around the sun.	Although the notion that the universe consisted of a system of nested spheres centered on the earth was still accepted as the nature of the universe, by the fifteenth century, Islamic and Jewish astronomers and Regiomontanus had noted discrepancies between Ptolemy's predictions and their own observations, and made adjustments to Ptolemy's details. By the Renaissance, it was recognized that the true solar year was $11\frac{1}{4}$ minutes less that the $365\frac{1}{4}$ days used to set the Julian calendar. Copernicus recognized that the various pieces of the Ptolemaic model simply couldn't be patched together to make all planets move as observed. Copernicus model is explained in his treatise *On the Revolutions of Celestial Spheres* of 1543 which collected the trigonometric background and computations as in the Almagest. The observed retrograde motion of a planet outside the orbit of the earth relative to the stars can be explained be the relative motions of the earth and the planet. To agree with observation, Copernicus also made planets move on epicycles of circles centered on the sun. The resulting theory is about as complicated as the Ptolemaic one.