Math 3070 § 1. $\alpha\tau$
Treibergs

Sample Second Exam

Solutions
February 22,2008

Second Midterm Exam Given Feb. 28, 2001.

(1) A Cache Valley cheese manufacturer claims that no more than 5% of all their packages contain less cheese than indicated on the label. To test this claim, 25 packages are randomly selected and weighed. The claim is accepted if fewer than 3 of the packages contain less cheese than indicated on the label. What is the probability that the claim is accepted if the actual percentage of packages with less cheese than indicated is 5%? 20%?

(2) Suppose that the continuous random variable X has a probability distribution function of the form

$$f(x) = \begin{cases} 0, & \text{if } x < 1; \\ 1/x, & \text{if } 1 \le x \le c; \\ 0, & \text{if } c < x. \end{cases}$$

Find the constant $c > 1$ which makes $f(x)$ a probability distribution. Find the cumulative distribution function of X. Find the probability that $c^{\frac{1}{3}} < X < c^{\frac{1}{2}}$. Find median of X (50th percentile.) Find mean of X. Find the variance of X.

(3) Suppose field mice are distributed at random in Juab County according to a Poisson Distribution with parameter $\alpha = 10$ per acre. What is the probability that there are at least six (6) but not more than nine (9) mice on a random acre? What is the expected number of mice that one would find on a random two (2) acre plot? How big should my sampling region be in order to be 90% sure of finding five (5) or more mice?

(4) A student commutes daily from his Murray home to The "U". The average time for a one way trip is 24 minutes with a standard deviation of 3.8 minutes. Assume that the duration of the trip is normally distributed. What is the probability that the trip will take at least 1/2 hour? Find the length of time such that only 15% of the trips take less than this time?

(5) Suppose on average, 1 person out of 1000 makes a numerical error preparing their Utah State Tax Return. If 10,000 returns are selected at random, find the probability that 6, 7 or 8 of the forms contain an error. Write an expression for the exact answer. Use an approximation to estimate your probability. Why is it acceptable to use this approximation?

Second Midterm Exam Given Oct. 13, 2004.

(1) Suppose that the proportion of blood phenotypes in Davis County is given by the table

A .41	B .11	AB .03	O .45

(a.) Assuming that the phenotypes of three randomly selected individuals from Davis County are mutually independent of one another, what is the probability that all three phenotypes are A? (b.) Still assuming independence, what is the probability that the phenotypes of three randomly selected individuals match?

(2) The probability density function (pdf) for the continuous random variable is given by $p(x) = 0$, if $x < 0$; .4, if $0 \le x < c$; .6, if $c \le x < 2$; 0, if $2 \le x$. where $0 < c < 2$ is some constant. (a.) Find c. (b.) Find the expectation $E(X)$. (c.) Find the cumulative distribution function (cdf) of X. (d.) Find the 90th percentile $\eta(.9)$ of X.

(3) Suppose that the dollar bill changer in the Olpin Union has a probability of 0.25 that it will not accept your bill to make change when you try to insert it. What is the probability that it takes you five or fewer trials to make change for three bills?

(4) The compressive strength of concrete for exhibition tanks has a population mean of 6000. psi and a standard deviation of 240.0 psi. Assume that the compressive strength is normally distributed. (a.) What is the probability that a random concrete sample have a compressive strength larger that 6228 psi? (b.) Below what psi will 20% of the samples of concrete be? [Answer to three decimal places.]

(5) The Castle Dale Candy Company claims that at most five of a shipment of twentyfive chocolate turkeys have a cream-filled center. To test this claim, four turkeys will be selected at random from the shipment and tasted. The claim is to be rejected if two or more turkeys are found to have a cream-filled center. What is the probability that the claim is rejected, given that the shipment actually contained five cream-filled turkeys?

(a.) What is the probability that the trip will take at least 1/2 hour? (b.) Find the length of time such that only 15% of the trips take less than this time?

The travel time in minutes X is assumed to be a normally distributed variable with mean $\mu = 24.$ and standard deviation $\sigma = 3.8$. Standardizing to Z-scores one finds $P(X \geq 30.) = 1 - P(X < 30) = 1 - P(Z = \frac{X-\mu}{\sigma} < \frac{30.-24.}{3.8} = 1.58) = 1 - \Phi(1.58) = 1 - .9429 = .0571.$ using the table on p. 740.

We seek a critical time $x_{.85}$ so that $P(X \leq x_{.85}) = .15$. We find the corresponding critical Z-value $P(Z \leq z_{.85}) = .15$. From p. 740: $\Phi(-1.03) = .1515$ and $\Phi(-1.04) = .1492$. The latter is closer to $.1500$ so, to two digits, $z_{.85} = -1.04$. Reverting to nonstandard variables, $x_{.85} = \mu + \sigma z_{.85} = (24.) + (3.8)(-1.04) = 20.0$ is the 15th-percentile critical trip time.

(5.) *Suppose on average, 1 person out of 1000 makes a numerical error preparing their Utah State Tax Return. If 10,000 returns are selected at random, find the probability that 6, 7 or 8 of the forms contain an error. Write an expression for the exact answer. Use an approximation to estimate your probability. Why is it acceptable to use this approximation?*

This is binomial random variable. The probability of success (a numerical error on one tax return) is $p = .001$. There are $n = 10,000$ returns to be checked and X is the number of these returns that have a numerical errors. Thus $P(6 \leq X \leq 8) = \text{Bin}(8; 10,000, .001) - \text{Bin}(5; 10,000, .001)$ where $\text{Bin}(x; n, p) = \sum_{y=0}^{x} \binom{n}{y} p^y (1-p)^{n-y}$.

We use the Poisson random variable to approximate. Thus the expected value $\lambda = np = (10,000)(.001) = 10$. Thus using p. 739, $P(6 \leq X \leq 8) \approx \text{Poisson}(8; 10) - \text{Poisson}(5; 10) = .333 - .067 = .266$.

This approximation is valid if n is large and p is small. According to the rule of thumb on p. 136 of the text, the approximation is acceptable if $n \geq 100$, $p \leq .01$ and $np \leq 20$. Here all conditions hold since $n = 10,000$, $p = .001$ and $np = 10$.

Solutions of the Fall '04 Midterm Problems

(1.)*Suppose that the proportion of blood phenotypes in Davis County is given by the table*

A .41	*B* .11	*AB* .03	*O* .45

(a.) Assuming that the phenotypes of three randomly selected individuals from Davis County are mutually independent of one another, what is the probability that all three phenotypes are A? (b.) Still assuming independence, what is the probability that the phenotypes of three randomly selected individuals match?

Let A_i be the event that the i-th individual has type 'A,' B_i be the event that the i-th individual has type 'B,' C_i be the event that the i-th individual has type 'AB,' D_i be the event that the i-th individual has type 'O.' By independence, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) = (.41)^3 = \boxed{0.068921}$. The event "all match" is all 'A' or all 'B' or all 'AB' or all 'O.' As these are mutually exclusive, $P(\text{all match}) = P(A_1 \cap A_2 \cap A_3) + P(B_1 \cap B_2 \cap B_3) + P(C_1 \cap C_2 \cap C_3) + P(D_1 \cap D_2 \cap D_3) = P(A_1)P(A_2)P(A_3) + P(B_1)P(B_2)P(B_3) + P(C_1)P(C_2)P(C_3) + P(D_1)P(D_2)P(D_3) = (.41)^3 + (.11)^3 + (.03)^3 + (.45)^3 = \boxed{.161404}$.

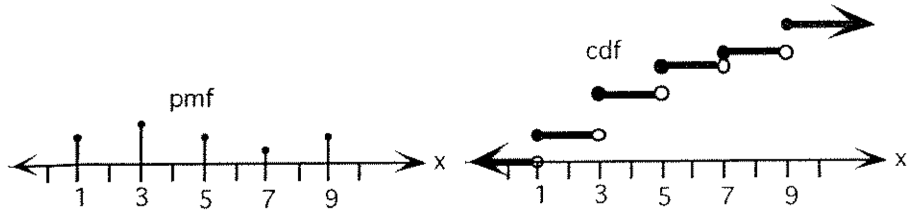
(2.)*The probability density function (pdf) for the continuous random variable is given by $p(x) = 0$, if $x < 0$; .4, if $0 \leq x < c$; .6, if $c \leq x < 2$; 0, if $2 \leq x$. where $0 < c < 2$ is some constant. (a.) Find c. (b.) Find the expectation $E(X)$. (c.) Find the cumulative distribution function (cdf) of X. (d.) Find the 90th percentile $\eta(.9)$ of X.*

The nonnegative function is a pdf if its total probability is one: $1 = \int_{-\infty}^{\infty} p(x)\,dx = \int_0^c (0.6)\,dx + \int_c^2 (0.4)\,dx = 0.6c + 0.4(2-c) = 0.8 + 0.2c$ so $c = \boxed{1}$. Then $E(X) = \int_{-\infty}^{\infty} x\,p(x)\,dx = \int_0^1 0.4x\,dx + \int_1^2 0.6x\,dx = 0.4(\frac{1}{2}) + 0.6(\frac{4-1}{2}) = \boxed{1.1}$. The cdf is $F(x) = \int_{-\infty}^{x} p(x)\,dx$. Thus $F(x) = 0$, if $x < 0$; $F(x) = .4x$, if $0 \leq x < 1$; $F(x) = .4 + .6(x-1) = 0.6x - 0.2$, if $c \leq x < 2$; $F(x) = 1$, if $2 \leq x$. The 90-th percentile is η so that $0.9 = F(\eta) = 0.6\eta - 0.2$ so $\eta = \frac{11}{6} = \boxed{1.83333}$.

(3.)*Suppose that the dollar bill changer in the Olpin Union has a probability of 0.25 that it will not accept your bill to make change when you try to insert it. What is the probability that it takes you five or fewer trials to make change for three bills?*

This is a negative binomial experiment. The random variable X is the number of failures before you get $r = 3$ successes. The probability of a "success" or "the changer takes your bill" is 0.75. Five or fewer bills inserted for three successes means 2 or fewer rejects, which is $P(X \leq 2) = \text{nb}(0, 3, 0.75) + \text{nb}(1, 3, 0.75) + \text{nb}(2, 3, 0.75) = \binom{2}{2}(0.75)^3(0.25)^0 + \binom{3}{2}(0.75)^3(0.25)^1 + \binom{4}{2}(0.75)^3(0.25)^2 = (0.75)^3[1 + 3 \cdot (0.25) + 6 \cdot (0.25)^2] = \boxed{0.896}$.

(4.)*The compressive strength of concrete for exhibition tanks has a population mean of 6000. psi and a standard deviation of 240.0 psi. Assume that the compressive strength is normally distributed. (a.) What is the probability*

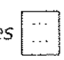


The cumulative disrtibution function is $F(x) = P(X \le x)$ is given by

$$F(x) = \sum_{\substack{j \in D \\ j \le x}} p(j) = \begin{cases} 0, & \text{if } x < 1; \\ .2, & \text{if } 1 \le x < 3; \\ .5, & \text{if } 3 \le x < 5; \\ .7, & \text{if } 5 \le x < 7; \\ .8, & \text{if } 7 \le x < 9; \\ 1.0, & \text{if } 9 \le x. \end{cases}$$

The probability $P(2 \le X \le 7) = P(\{X = 3\} \cup \{X = 5\} \cup \{X = 7\}) = p(3) + p(5) + p(7) = .3 + .2 + .1 = .6$.
The expectation $E(X) = \sum_{x \in D} x p(x) = 1(.2) + 3(.3) + 5(.2) + 7(.1) + 9(.2) = 4.6$.
The expectation of the squares is $E(X^2) = \sum_{x \in D} x^2 p(x) = 1^2(.2) + 3^2(.3) + 5^2(.2) + 7^2(.1) + 9^2(.2) = 29.0$.
The standard deviation, by the short cut formula satisfies $\sigma^2 = E(X^2) - [E(X)]^2 = 29.00 - 21.16 = 7.84$ so $\sigma = 2.80$. Since R is a linear function, using the transformation of the expected value yields

$E(R) = E(8 - (.5)X) = 8.00 - (.5)E(X) = 8.00 - (.5)(4.6) = 5.70$.

(4.) *Suppose a fair 6-sided die is rolled five times. Let X denote the number of sixes*  *in the five rolls. What is the probability that $X = 3$? What is the probability of $X \ge 3$?*

This is a binomial random variable with $n = 5$ trials and probability of a success (hitting a six) is $p = 1/6$. This value of p is not tabulated, so we compute it. Thus $P(X = 3) = \text{bin}(3; 5, 1/6) = \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = .0322$.
The probability $P(X \ge 3) =$

$\text{bin}(3; 5, 1/6) + \text{bin}(4; 5, 1/6) + \text{bin}(5; 5, 1/6) = \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 + \binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1 + \binom{5}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0 = .0355$.

(5.) *A food processor claims that at most 10% of her jars of instant drink powder contain less powder than claimed on the label. To test this claim, 16 jars are randomly selected and the contents are weighed. Her claim is accepted if fewer than 3 of the jars contain less powder than claimed on the label. Find the probability that the food processors claim will be accepted when the actual percentage of jars containing less powder than claimed is 10%? 20%?*

This is a binomial variable. x is the number of successes (underweight jars) in $n = 16$ selected. We wish to compute the probability that the claim is accepted, in other words fewer than three, *i.e.* 2 or less jars are underweight. This is the cdf $P(X \le 2) = \text{bin}(0; 16, p) + \text{bin}(1; 16, p) + \text{bin}(2; 16, p)$. ($n = 16$ is not tabulated in the text so we must compute.)

$P(X \le 2) = \binom{16}{0}p^0(1-p)^{16} + \binom{16}{1}p^1(1-p)^{15} + \binom{16}{2}p^2(1-p)^{14}$.

For $p = .1$, $P(X \le 2) = .789$. For $p = .2$, $P(X \le 2) = .352$.

(6.) *A shipment of 120 burglar alarms contains 6 that are defective. If three alarms are selected randomly and shipped to the customer, what is the probability that the customer will get one defective unit using the formula for the hypergeometric distribution? One or more defective units? Using the binomial distribution as approximation? Does this example satisfy the text's rule of thumb when binomial may be used to approximate hypergeometric?*

First analyze the hypergeometric. The number of successes (defective alarms) is $M = 6$ in a the total $N = 120$. $n = 3$ are selected without replacement and X is the number of successes in the selection. We wish to compute the probability that $X = 1$. This is $P(X = 1) = \text{hyp}(1; 3, 6, 120)$. The probability that at least one alarm is defective is $P(X \ge 1)$.

$P(X = 1) = \frac{\binom{6}{1}\binom{114}{2}}{\binom{120}{3}} = .138 \qquad P(X \ge 1) = 1 - P(X = 0) = 1 - \frac{\binom{6}{0}\binom{114}{3}}{\binom{120}{3}} = .144.$

Now the binomial approximation. $p = M/N = 6/120 = .05$. Then $P(X = 1) = \text{bin}(1; 3, .05)$ and $P(X \ge 1) = 1 - P(X = 0)$ so

$P(X = 1) = \binom{3}{1}(.05)^1(.95)^2 = .135, \qquad P(X \ge 1) = 1 - \binom{3}{0}(.05)^0(.95)^3 = .143.$

The rule of thumb, according to p. 122 of the text is that if $n/N \le .05$ then sampling without replacement

Finally $P(X > \pi/4) = 1 - P(X \le \pi/4) = 1 - F(\pi/4) = 1 - \frac{1-\sqrt{2}/2}{2} = .854$.

(10.)*Suppose that cumulative distribution of X is $F(x) = 0$, if $x < 0$; $F(x) = 2x^2 - x^3$, if $0 \le x \le 1$; $F(x) = 1$, if $x > 1$. Find the 37.5th percentile of distribution. Find the mean and standard deviation.*

The 37.5% of the distribution is the point η where $P(X \le \eta) = F(\eta) = .375 = 3/8$. Because F is strictly increasing on $(0,1)$ since $f' > 0$, there can only be one number η which solves the equation $F(\eta) = 3/8$. Solve the equation by trial and error. One checks that $F(1/2) = 2/4 - 1/8 = 3/8$ so $\eta = 1/2$. To find the mean and standard deviation we need the probability distribution function. The fundamental theorem of calculus relates the probability distribution function to the cumulative distribution function $F'(x) = f(x)$. Thus differentiating

$$f(x) = \begin{cases} 0, & \text{if } x < 0; \\ 4x - 3x^2, & \text{if } 0 \le x \le 1; \\ 0, & \text{if } 0 < x. \end{cases}$$

Then the mean and expected square is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 4x^2 - 3x^3\,dx = \left[\frac{4x^3}{3} - \frac{3x^4}{4}\right]_0^1 = \left[\frac{4}{3} - \frac{3}{4}\right] = \frac{7}{12},$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 4x^3 - 3x^4\,dx = \left[x^4 - \frac{3x^5}{5}\right]_0^1 = 1 - \frac{3}{5} = \frac{2}{5}.$$

Finally, the standard deviation $\sigma = .244$ since $\sigma^2 = E(X^2) - [E(X)]^2 = \frac{2}{5} - \frac{49}{144} = \frac{43}{720}$.

(11.) *Assume development time for a particular photographic print paper which was exposed by an enlarger for 5 seconds is normally distributed with a mean of 28.0 seconds and a standard deviation of 1.40 seconds. What is the probability that a print will take more than 29.5 sec. to develop? That the development time differs from the expected by more than 2.50 seconds?*

We are given that the development time X is normally distributed with mean $\mu = 28.0$ and standard deviation $\sigma = 1.40$. The probability is read from the table on p. 741, after converting to Z and interpolating, and using symmetry of the normal distribution, $P(X > 29.5) = 1 - P(X \le 29.5) =$

$1 - P\left(Z = \frac{X-\mu}{\sigma} \le \frac{29.5-28.0}{1.4} = 1.071\right) = 1 - \Phi(1.071) = 1 - 0.8579 = 0.1421$.

$P(X > 30.5 \text{ or } X < 25.5) = 2P(X < 25.5) = 2P\left(Z = \frac{X-\mu}{\sigma} < \frac{25.5-28.0}{1.4} = -1.786\right) = 2\Phi(-1.786) = 2(0.0370) = .0740$.

(12.) *Suppose when the outside temperature is μ, the readings from a thermometer are normally distributed with mean μ and standard deviation σ. What would σ have to be to ensure that 95% of all readings are within $.5^\circ$ of μ?*

If X is the reading from the thermometer, we are given that X is normally distributed with mean μ and standard deviation σ. Standardizing to $Z \sim N(0,1)$, we need to know the value z_α so that the region from $-z_\alpha \le Z \le z_\alpha$ accounts for 95% of the area under the curve. In other words, the neglected tails of the distribution both account for 5% of the area, or each accounts for 2.5%. *i.e.* $.95 = P(-z_\alpha \le Z \le z_\alpha) = 1 - 2P(Z > z_\alpha)$ implies $P(Z > z_\alpha) = (1 - .95)/2 = .025$. Thus $\alpha = .025$ and the critical value $z_\alpha = 1.96$ is gotten from Table 4.1 (p. 165). Thus, we solve for σ so that z_α corresponds to $.5^\circ$. $.5 = x_\alpha - \mu = \sigma z_\alpha = \sigma(1.96)$ which implies that the desired $\sigma = 0.255^\circ$.

(13.) *It is claimed that more than 40% of the households in the Salt Lake City metro area subscribe to the newspaper* Deseret News. *A test of this hypothesis is conducted by telephone polling 20 random households in the metro area. The claim is accepted if at least six households in the poll subscribe to the* Deseret News. *What is the probability that the survey would not accept the claim even if 40% of households in the area actually do subscribe to* Deseret News? *Suppose the poll is enlarged to 200 households instead. What is the probability that fewer than 75 responding subscribe to the Deseret News if in fact 40% of the households in the area actually do subscribe to the* Deseret News? *Use an approximation. Is your approximation justified by the text's rule of thumb?*

This is a binomial variable and the probability of a single success (subscribing to *Deseret News*) is $p = .4$. If there are $n = 20$ trials and $x =$ the number of successes, we are asked what is the probability that the claim is rejected. That is, what is $P(x \le 5) = \sum_{x=0}^{5} \text{bin}(x; 20, .4) = 0.1256$ from Table A.1 on p. 737.