Math 3070 § 1.	**Blade Stress Example: Estimate**	Name: Example
Treibergs	**Parameter of Rayleigh Distribution.**	June 15, 2011

Problem 4.4 and 6.16 of Devore discuss the Rayleigh Distribution, whose pdf is

$$f(x;\theta) = \begin{cases} \dfrac{x}{\theta}\exp\left(-\dfrac{x^2}{2\theta}\right), & x > 0; \\ 0. & x \le 0. \end{cases}$$

The expectation of X^2 is found by substituting $p = x^2/(2\theta)$ so $dp = x/\theta\,dx$ and $\dfrac{d}{dp}\left(-(1+p)e^{-p}\right) = pe^{-p}$ so

$$\begin{aligned} \mathrm{E}(X^2) &= \int_{x=0}^{\infty} x^2 f(x;\theta)\,dx \\ &= \int_{x=0}^{\infty} \frac{x^3}{\theta} e^{-x^2/2\theta}\,dx \\ &= 2\theta \int_{p=0}^{\infty} pe^{-p}\,dp \\ &= 2\theta\Big[-(1+p)e^{-p}\Big]_{p=0}^{p=\infty} \\ &= 2\theta. \end{aligned}$$

Hence for a random sample $X_1,\ldots,X_n$ from this distribution,

$$\mathrm{E}\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right) = \theta.$$

Thus an unbiased estimator for θ is

$$\hat\theta = \frac{1}{2n}\sum_{i=1}^{n} X_i^2.$$

We apple the estimate to the data from the article "Blade Fatigue Life…" *J. Solar Energy*, 1982, to get an estimte of θ and superimpose the pdf onto the density histogram.

We also find a way to generate Rayleigh random variables. Suppose that the cdf is $F(x) = \mathrm{P}(X \le x)$, which is strictly increasing on $x > 0$. Then if $U \sim \mathrm{Unif}(0,1)$ is a standard uniform RV then $Y = F^{-1}(U)$ is a Rayliegh RV. To tee this,

$$\mathrm{P}(Y \le y) = \mathrm{P}\left(F^{-1}(U)) \le y\right) = \mathrm{P}\left(U \le F(y)\right) = F(y)$$

because, for numbers $0 \le m \le 1$, for uniform variables, $\mathrm{P}(U \le m) = m$. Thus the simulation can be carried out because we can find $F(x)$ for Rayleigh RV's.

$$\begin{aligned} F(x) &= \int_0^x f(x;\theta)\,dx \\ &= \int_0^x \frac{x}{\theta} e^{-x^2/2\theta}\,dx \\ &= \int_{p=0}^{x^2/(2\theta)} e^{-p}\,dp \\ &= \Big[-e^{-p}\Big]_{p=0}^{p=x^2/(2\theta)} \\ &= 1 - e^{-x^2/(2\theta)}. \end{aligned}$$

Finally, we observe that a Rayleigh RV is a special case of the Weibull distribution. $W \sim \text{Weibull}(\alpha, \beta)$ has a pdf

$$\text{dweibull}(x; \alpha, \beta) = \begin{cases} \dfrac{\alpha}{\beta^\alpha} x^{\alpha-1} \exp\left(-\left(\dfrac{x}{\beta}\right)^\alpha \right), & x > 0; \\ 0. & x \le 0. \end{cases}$$

Thus by setting $\alpha = 2$ and $\beta = \sqrt{2\theta}$, we see that

$$f(x; \theta) = \text{dweibull}(x, 2, \sqrt{2\theta}).$$

R Session:

```
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[R.app GUI 1.34 (5589) i386-apple-darwin9.8.0]

[Workspace restored from /home/1004/ma/treibergs/.RData]

> ######################### DOWLOAD A BUNCH OF COLORS ##########################
> library(RColorBrewer)
> colr <- c(brewer.pal(9,"Set1"),brewer.pal(8,"Set2"))
>
> ######################### ENTER THE RAYLEIGH PDF #############################
> f <- function(x,theta){ x*exp(-x^2/(2*theta))/theta}
>
> xs <- seq(0,10,.099)
> # Maximum of f(x,1) at  fmax. for j > 1  maximum of f(x,j) is less than  fmax.
> fmax <- exp(-.5)
```
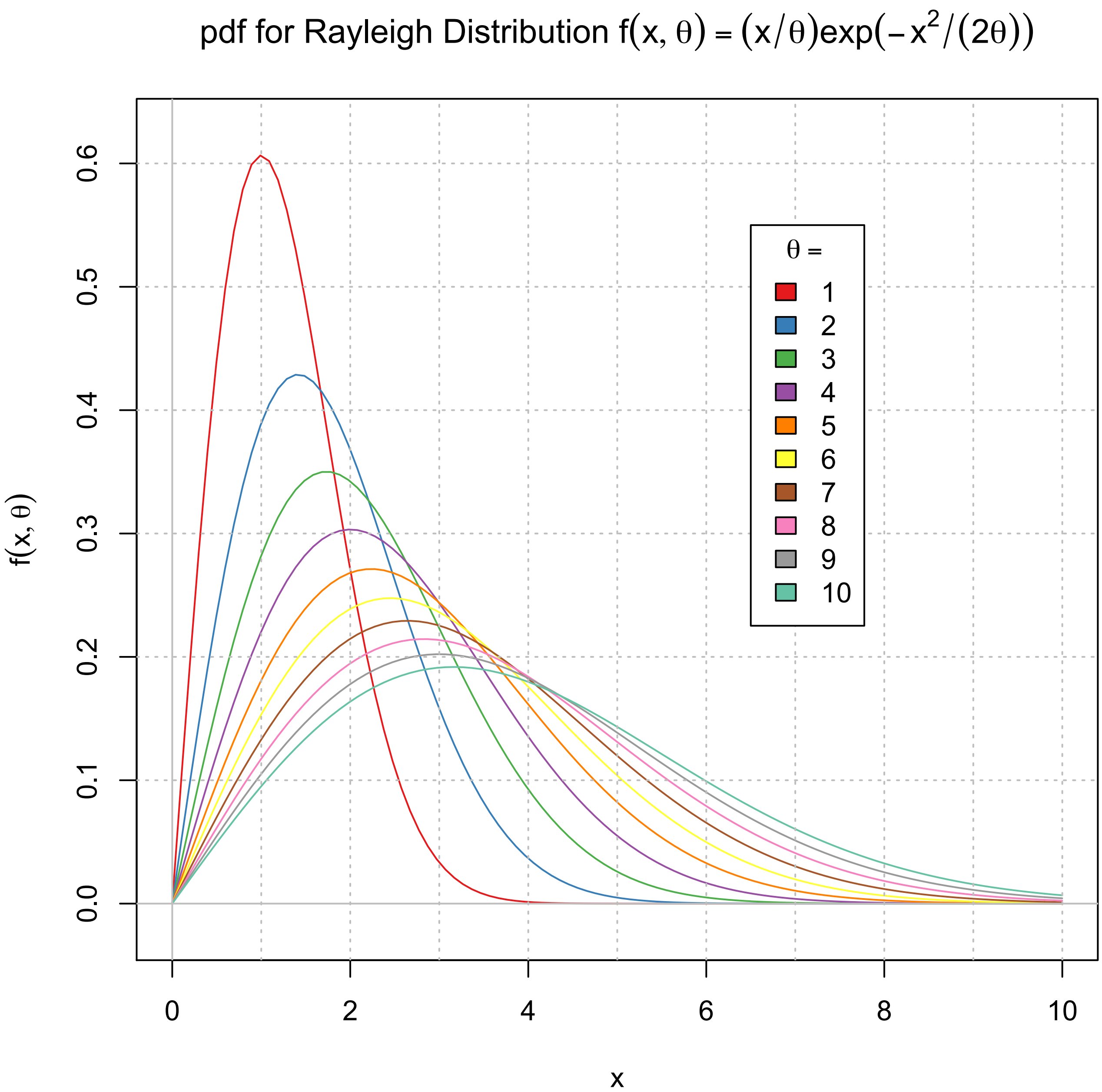
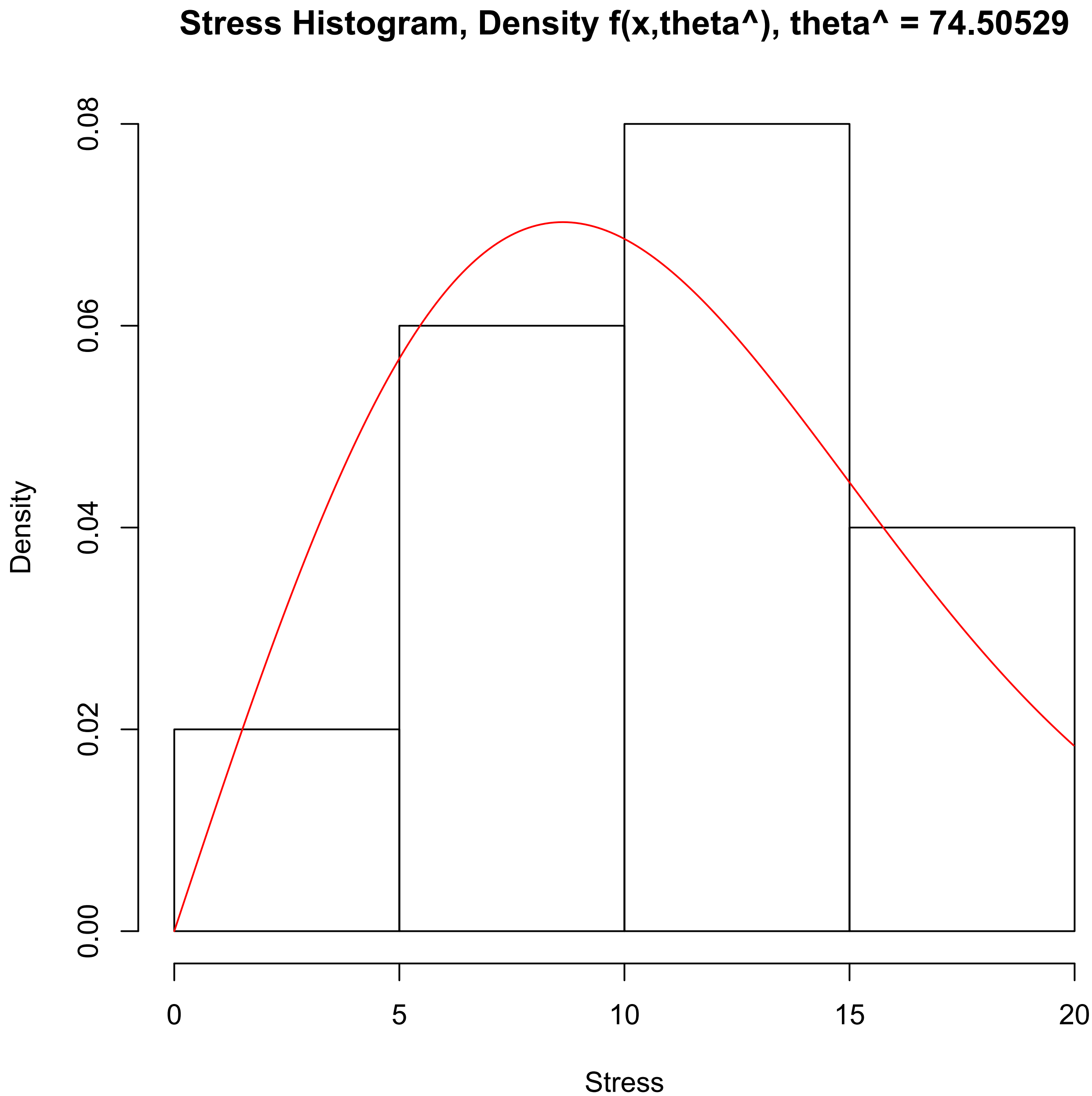

```
> plot(xs,f(xs,1), type="n", ylab=expression(f(x,theta)), xlab="x",
+ main=expression(paste("pdf for Rayleigh Distribution ",
+ f(x,theta)==(x/theta)*exp(-x^2/(2*theta)))), ylim=c(-.02,fmax+.02))
> for(j in 1:10)
+               {
+                lines(xs, f(xs,j), col = colr[j])
+               }
> abline(h=0,col=8);abline(v=0,col=8)
> abline(h=(1:6)/10,col=8,lty=3);abline(v=1:10,col=8,lty=3)
> legend(6.5,.55, legend = 1:10, fill = colr[1:10], bg="white",
+ title=expression(theta==""))
> # M3074Rayleigh1.pdf
```



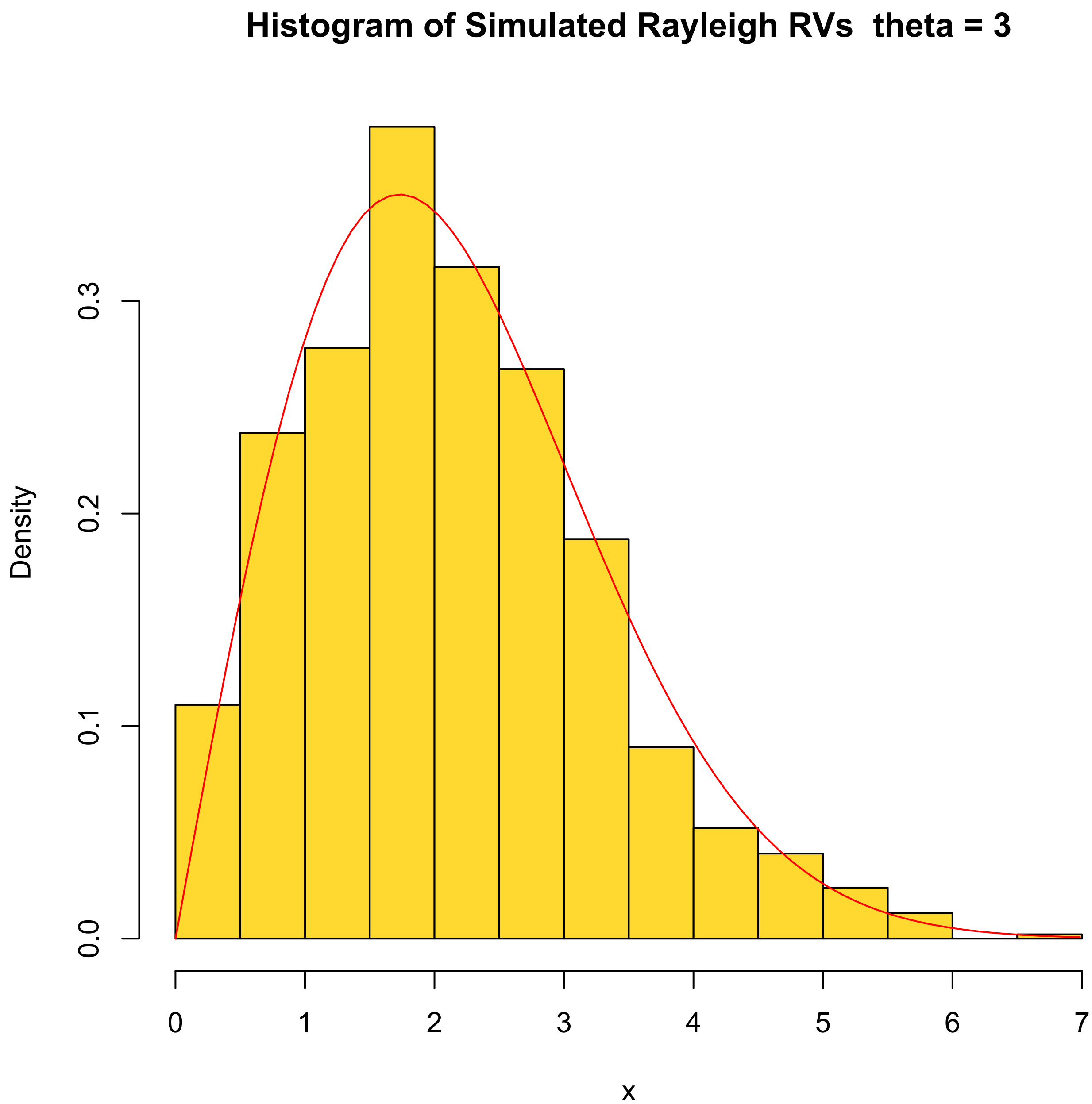

```
> ################ ENTER VIBRATION DATA #########################################
> # Devore 6.15  Vibratory Stress of turbine Blade from
> # ‘‘Blade Fatigue Life\ldots’’ {\it J.~Solar Energy}, 1982
> Stress <- scan()
1: 16.88 10.23 4.59 6.66 13.68
6: 14.23 19.87 9.40 6.51 10.95
11:
Read 10 items
```

```
> Stress
 [1] 16.88 10.23  4.59  6.66 13.68 14.23 19.87  9.40  6.51 10.95
>
> ################## ESTIMATOR THETA HAT ##########################################
> thetahat <- .5*mean(Stress^2)
> # Plot histogram of "Stress" and density using estimated "thetahat"
> hist(Stress, breaks = "FD", freq = FALSE,
+ main = paste("Stress Histogram, Density f(x,theta^), theta^ =", thetahat))
> lines(xxs, f(xxs,thetahat), col=2)
> # M3074Rayleigh2.pdf
```



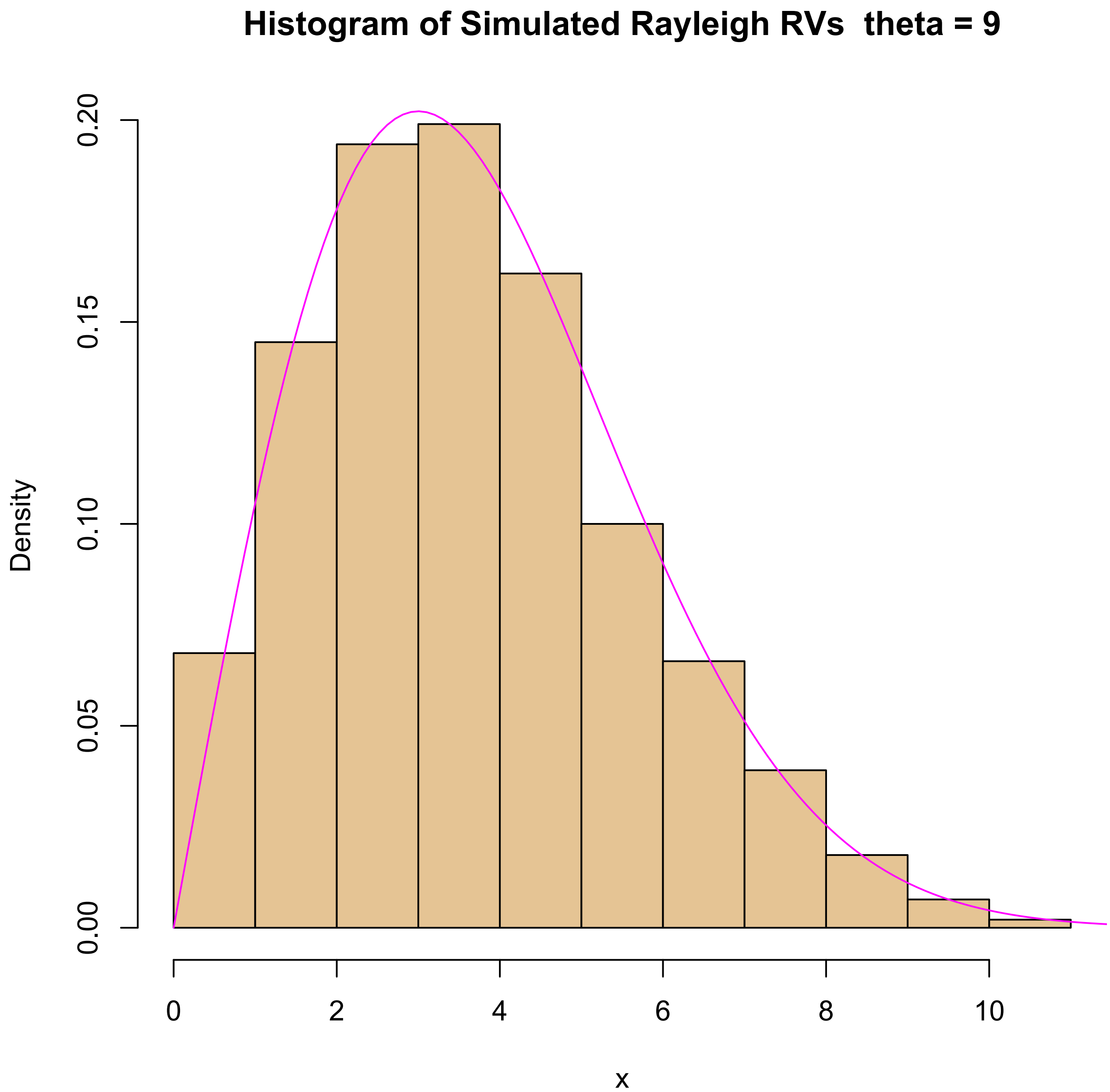

```
> ##################### SIMULATING Rayleigh rv's ################################
> # method works using cdf
> #  F(x) = P(X <= x).
> # F^(-1)(p) = inverse of F
> th3 <- 3
> Finv <- function(p){sqrt(-2*th3*log(1-p))}
> # sapply( numbers, function ) evaluates the "function"  on each of the "numbers."
> hist(sapply(runif(1000),"Finv"), xlab="x", freq = FALSE,
+ main = paste("Histogram of Simulated Rayleigh RVs  theta =", th3), col=colr[15])
```

```
> xxxs<-seq(0,7,.097)
> lines(xxxs,f(xxxs,th3),col=2)
> # M3074Rayleigh3.pdf
```



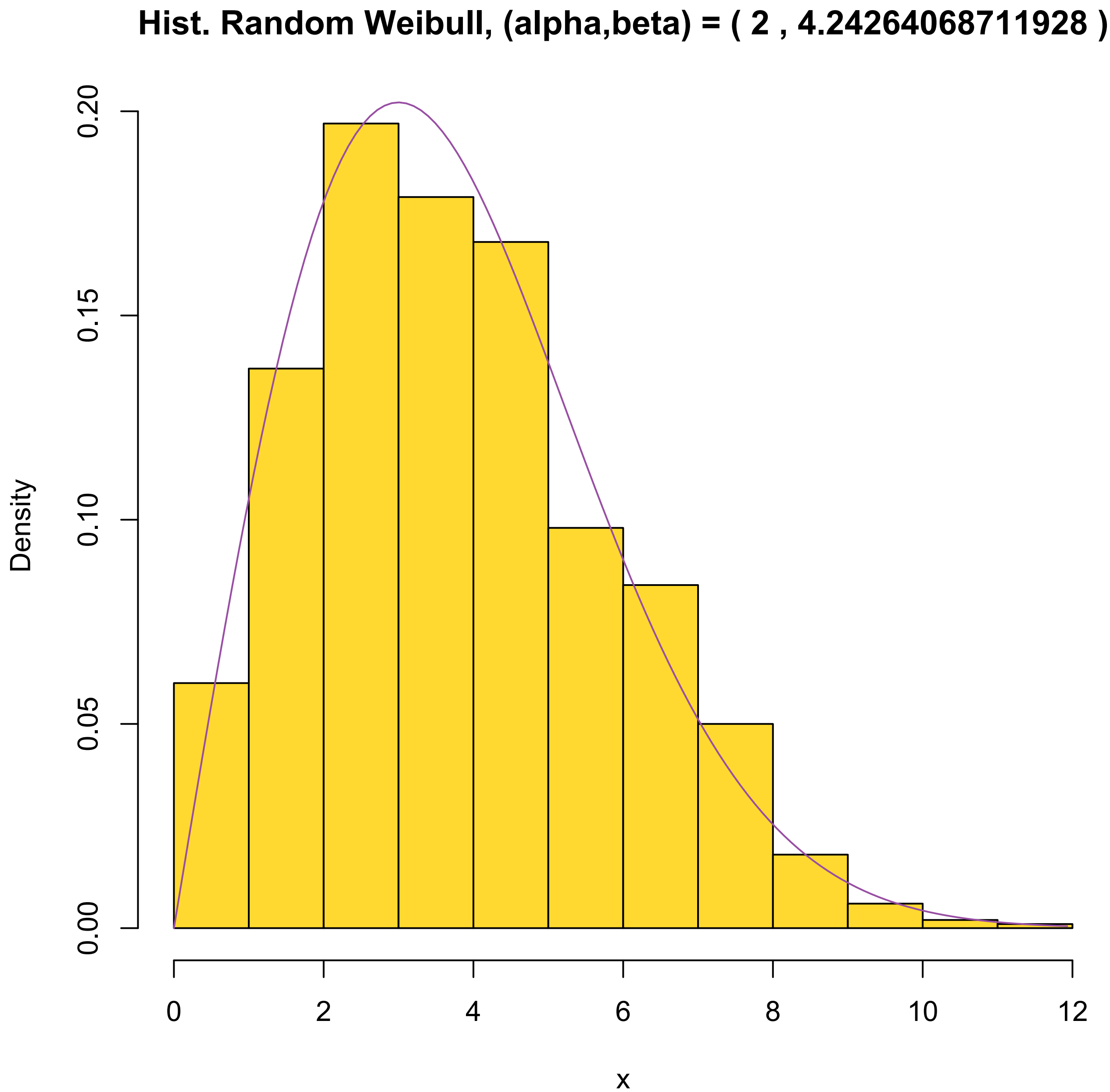

```
> th3 <- 9
> Finv <- function(p){sqrt(-2*th3*log(1-p))}
> xxxs<-seq(0,12,.097)
```

```
> hist(sapply(runif(1000),"Finv"), xlab="x", freq=FALSE,
+ main = paste("Histogram of Simulated Rayleigh RVs  theta =", th3), col=colr[16])
> lines(xxxs, f(xxxs,th3), col = 6)
> # M3074Rayleigh4.pdf
```




```
> ###################### TEST IF RAYLEIGH EQUALS WEIBULL ###########################
> # Compute difference  f(x,theta) - dweibull(x,2,sqrt(2*theta)  for several x,theta
> s <- matrix(numeric(50), ncol=5, dimnames = list((1:10)/2, 1:5))
```

```
>
> for(i in 1:10)
+             {
+              for(j in 1:5)
+                          {
+                           s[i,j]<-f(i/2,j)-dweibull(i/2,2,sqrt(2*j))
+                          }
+             }
> s
               1 2             3             4             5
0.5  5.551115e-17 0 -5.551115e-17  2.775558e-17  0.000000e+00
1    0.000000e+00 0  0.000000e+00  2.775558e-17  0.000000e+00
1.5  0.000000e+00 0  0.000000e+00  0.000000e+00  0.000000e+00
2   -5.551115e-17 0  0.000000e+00  0.000000e+00  0.000000e+00
2.5 -1.387779e-17 0 -5.551115e-17  0.000000e+00  0.000000e+00
3   -2.775558e-17 0  0.000000e+00  0.000000e+00 -2.775558e-17
3.5  0.000000e+00 0  2.775558e-17  2.775558e-17  2.775558e-17
4   -2.168404e-18 0  4.163336e-17 -2.775558e-17  0.000000e+00
4.5  0.000000e+00 0 -6.938894e-18  1.387779e-17 -2.775558e-17
5   -3.049319e-20 0 -6.938894e-18 -6.938894e-18 -1.387779e-17

> # Hmmmmmm! Very small differences!
>
> ############### USE WEIBULL TO SIMULATE RAYLEIGH RV ####################
> hist( rweibull(1000,2,sqrt(2*th3)), xlab = "x", freq = FALSE,
+ main=paste("Hist. Random Weibull, (alpha,beta) = (", 2, "," , sqrt(2*th3),
+ ")"),col=colr[15])
> lines(xxxs, dweibull(xxxs, 2, sqrt(2*th3)), col = colr[4])
> # M3074Rayleigh5.pdf
```


Hist. Random Weibull, (alpha,beta) = (2 , 4.24264068711928)
Density
0.00
0.05
0.10
0.15
0.20
0
2
4
6
8
10
12
x