Math 3070 § 1. **Oxygen Consumption Example: Hodges-Lehman** Name: Example
Treibergs **Estimator for Center. Bootstrap & Simulation.** July 8, 2011

The Hodges-Lehman estimator is introduced in Problem 7.35 of Devore, *Probability and Statistics for Engineering and the Sciences,* 8*th ed.,* Brooks Cole, 2012. Devore applies it to data from the article "Oxygen Consumption During Fire Supression: Error of Heart Rate Estimation," *Ergonomics,* 1991. The sample gives oxygen consumption in (mL/kg/min) for ten firefighters.

The Hodges-Lehmann Estimator is the median of pairwise averages of all observations. It is known to be a good estimator for the center of a symmetric distribution. We apply it to the fireman data, and do simulations to assess its standard error by a bootstrap procedure. We take $B = 10,000$ samples of size $n = 10$ with replacement from the data (as a proxy for the background distribution) and compute the Hodges-Lehman for each. The bootstrapped standard error is $S_{\hat{\mu}} = 1.715$.

We also take $B = 10,000$ samples of size $n = 10$ from t-distribution with 5 degrees of freedom. Its variance is $\sigma^2 = \alpha/(\alpha - 2) = 5/(5 - 2) = 1.667$ so $\sigma = 1.291$. The simulated standard error of the Hodges-Lehman estimator is in this example $S_{\hat{\mu}} = .376$.

We use **R**'s vector arithmetic. Let a sample $X_,.X_2, \ldots, X_n$, be taken from a pdf symmetric about μ. For each $1 \le i \le j \le n$ we compute

$$\bar{X}_{i,j} = \frac{X_i + X_j}{2},$$

the mean of two observations. Then the Hodges-Lehman estimator is

$$\hat{\mu} = \operatorname*{median}_{i \le j} \bar{X}_{i,j}.$$

To compute it, `.5*outer(x,x,"+")` makes an array of pairwise averages. The statistic is the median of the upper triangle including the diagonal. The median of the whole matrix double counts off-diagonal entries. By throwing in another diagonal, we get adouble count of all entries, thus has the same median

$$\hat{\mu} = \text{median}\left(\left\{\bar{X}_{i,j} : 1 \le i, j \le n\right\} \cup \left\{\bar{X}_{i,i} : 1 \le i \le n\right\}\right).$$

R Session:

```
R version 2.10.1 (2009-12-14)
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ISBN 3-900051-07-0

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Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

[R.app GUI 1.31 (5538) powerpc-apple-darwin8.11.1]
[Workspace restored from /Users/andrejstreibergs/.RData]
```
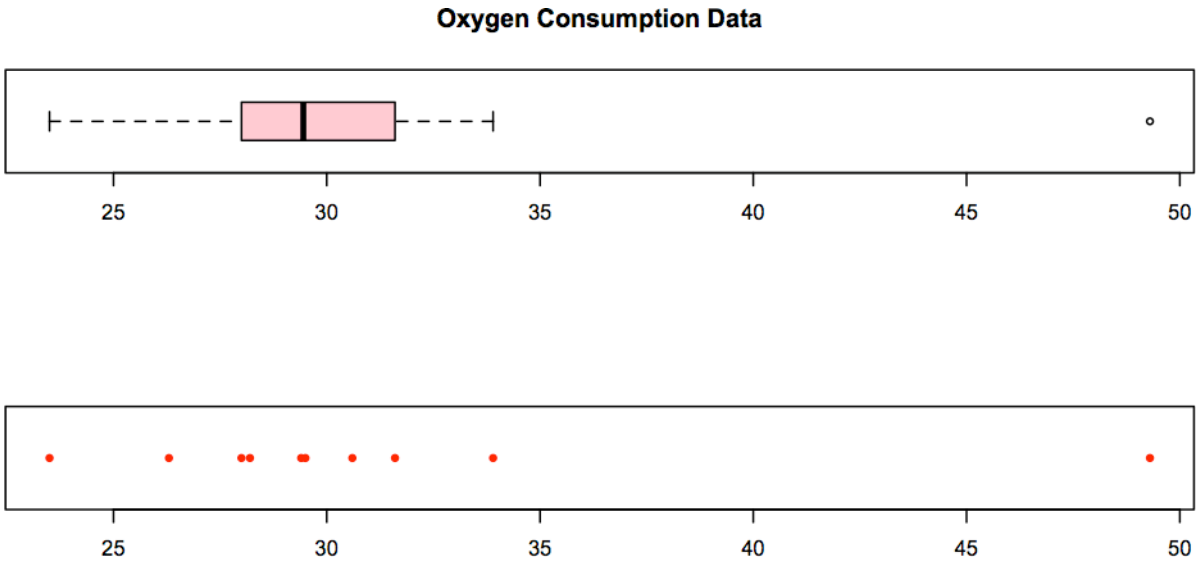
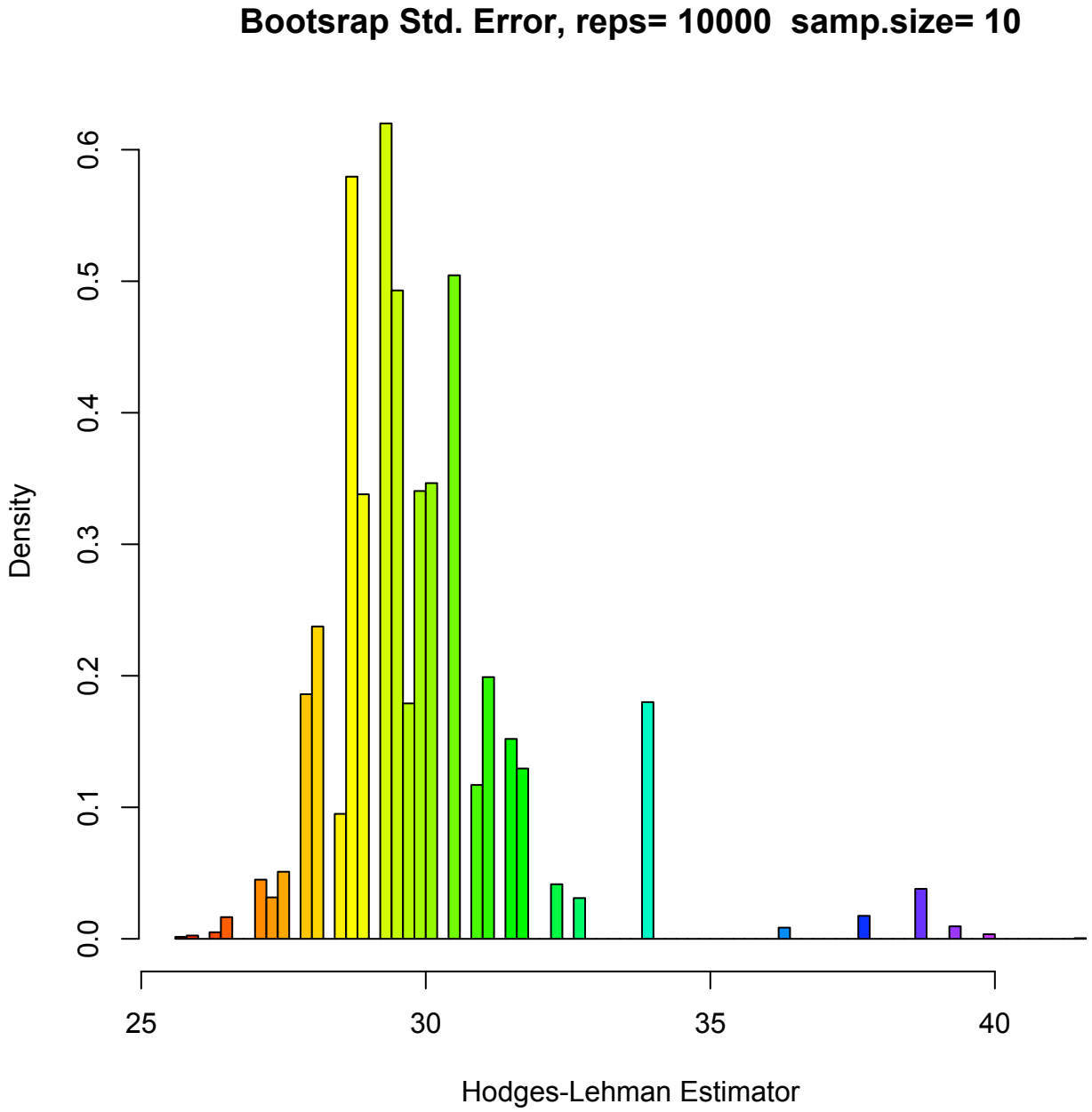
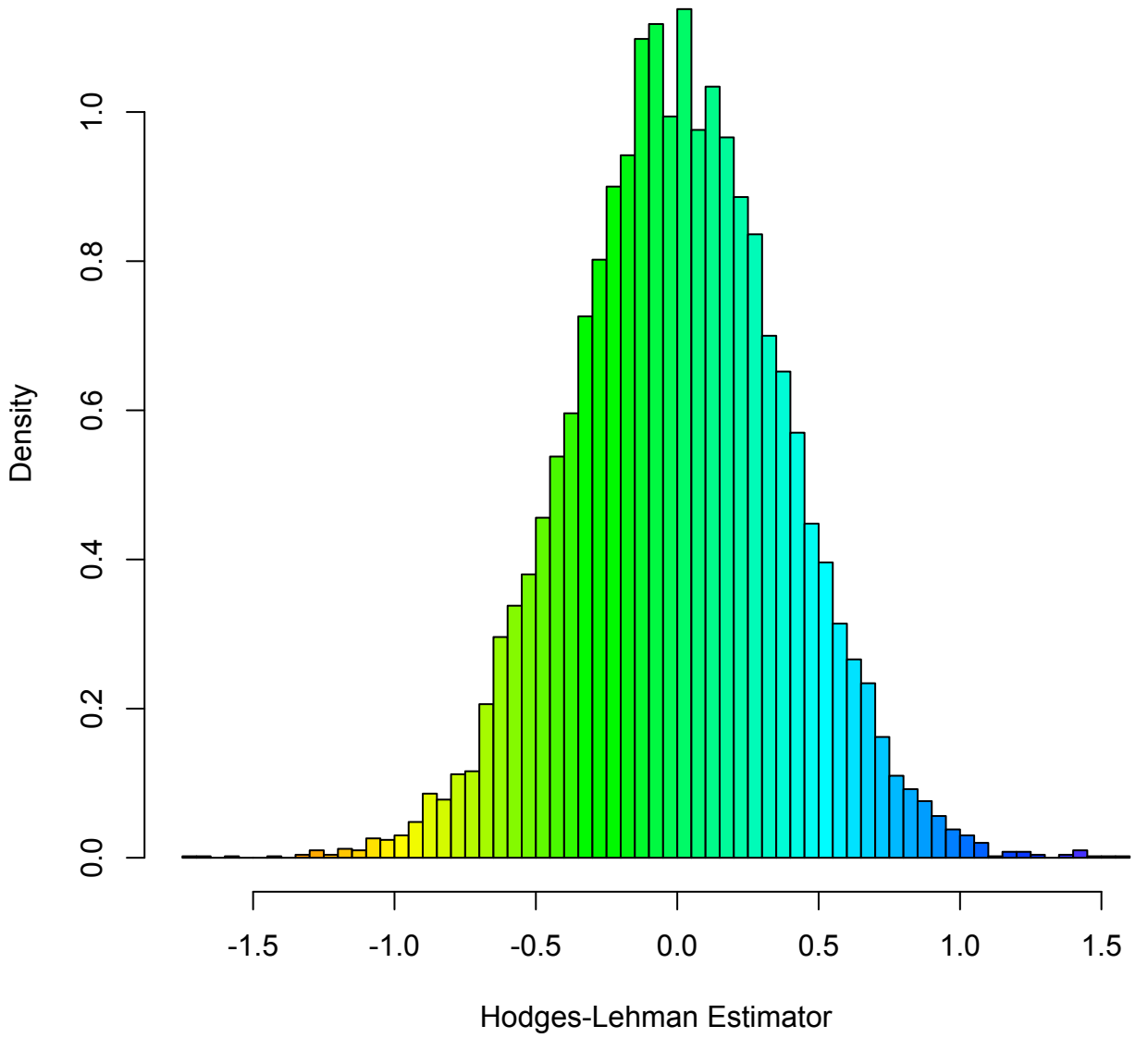

```
> ###################### ENTER OXY CONS DATA ###################################
> # Devore 1.44
> x <- scan ()
1: 29.5 30.6 49.3 28.2 28.0 26.3 33.9 29.4 23.5 31.6
11:
Read 10 items
> x
 [1] 29.5 30.6 49.3 28.2 28.0 26.3 33.9 29.4 23.5 31.6
> summary(x)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  23.50   28.05   29.45   31.03   31.35   49.30
> range(x)
[1] 23.5 49.3
> # Find sample range.
> max(x)-min(x)
[1] 25.8

> # Find variance and s.d.
> var(x); sd(x)
[1] 49.31122
[1] 7.022195

> # plot boxplot sand stripchart of this data.
> boxplot(x,horizontal=T,col="pink")
> title("Oxygen Consumption Data")
> stripchart(x,method="stack",pch=19,col=2)
> # M3074OxyCons1.pdf
```



```
> ################### FUNCTION TO COMPUTE HODGES-LEHMAN STATISTIC ############
> # devore 6.35  Hodges lehman estimator
> hl <- function(x){ median(c(.5*outer(x, x, "+"), x)) }
> hl(x)
[1] 29.5
> ############### TO UNDERSTAND THE CODE, WE DISSECT IT: ######################
> outer(x,x,"+")
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
 [1,] 59.0 60.1 78.8 57.7 57.5 55.8 63.4 58.9 53.0  61.1
 [2,] 60.1 61.2 79.9 58.8 58.6 56.9 64.5 60.0 54.1  62.2
 [3,] 78.8 79.9 98.6 77.5 77.3 75.6 83.2 78.7 72.8  80.9
 [4,] 57.7 58.8 77.5 56.4 56.2 54.5 62.1 57.6 51.7  59.8
 [5,] 57.5 58.6 77.3 56.2 56.0 54.3 61.9 57.4 51.5  59.6
 [6,] 55.8 56.9 75.6 54.5 54.3 52.6 60.2 55.7 49.8  57.9
 [7,] 63.4 64.5 83.2 62.1 61.9 60.2 67.8 63.3 57.4  65.5
 [8,] 58.9 60.0 78.7 57.6 57.4 55.7 63.3 58.8 52.9  61.0
 [9,] 53.0 54.1 72.8 51.7 51.5 49.8 57.4 52.9 47.0  55.1
[10,] 61.1 62.2 80.9 59.8 59.6 57.9 65.5 61.0 55.1  63.2
> .5*outer(x,x,"+")
       [,1]  [,2]  [,3]  [,4]  [,5]  [,6]  [,7]  [,8]  [,9] [,10]
 [1,] 29.50 30.05 39.40 28.85 28.75 27.90 31.70 29.45 26.50 30.55
 [2,] 30.05 30.60 39.95 29.40 29.30 28.45 32.25 30.00 27.05 31.10
 [3,] 39.40 39.95 49.30 38.75 38.65 37.80 41.60 39.35 36.40 40.45
 [4,] 28.85 29.40 38.75 28.20 28.10 27.25 31.05 28.80 25.85 29.90
 [5,] 28.75 29.30 38.65 28.10 28.00 27.15 30.95 28.70 25.75 29.80
 [6,] 27.90 28.45 37.80 27.25 27.15 26.30 30.10 27.85 24.90 28.95
 [7,] 31.70 32.25 41.60 31.05 30.95 30.10 33.90 31.65 28.70 32.75
 [8,] 29.45 30.00 39.35 28.80 28.70 27.85 31.65 29.40 26.45 30.50
 [9,] 26.50 27.05 36.40 25.85 25.75 24.90 28.70 26.45 23.50 27.55
[10,] 30.55 31.10 40.45 29.90 29.80 28.95 32.75 30.50 27.55 31.60
> c(.5*outer(x,x,"+"),x)
  [1] 29.50 30.05 39.40 28.85 28.75 27.90 31.70 29.45 26.50 30.55 30.05 30.60
 [13] 39.95 29.40 29.30 28.45 32.25 30.00 27.05 31.10 39.40 39.95 49.30 38.75
 [25] 38.65 37.80 41.60 39.35 36.40 40.45 28.85 29.40 38.75 28.20 28.10 27.25
 [37] 31.05 28.80 25.85 29.90 28.75 29.30 38.65 28.10 28.00 27.15 30.95 28.70
 [49] 25.75 29.80 27.90 28.45 37.80 27.25 27.15 26.30 30.10 27.85 24.90 28.95
 [61] 31.70 32.25 41.60 31.05 30.95 30.10 33.90 31.65 28.70 32.75 29.45 30.00
 [73] 39.35 28.80 28.70 27.85 31.65 29.40 26.45 30.50 26.50 27.05 36.40 25.85
 [85] 25.75 24.90 28.70 26.45 23.50 27.55 30.55 31.10 40.45 29.90 29.80 28.95
 [97] 32.75 30.50 27.55 31.60 29.50 30.60 49.30 28.20 28.00 26.30 33.90 29.40
[109] 23.50 31.60
> sort(c(.5*outer(x,x,"+"),x))
  [1] 23.50 23.50 24.90 24.90 25.75 25.75 25.85 25.85 26.30 26.30 26.45 26.45
 [13] 26.50 26.50 27.05 27.05 27.15 27.15 27.25 27.25 27.55 27.55 27.85 27.85
 [25] 27.90 27.90 28.00 28.00 28.10 28.10 28.20 28.20 28.45 28.45 28.70 28.70
 [37] 28.70 28.70 28.75 28.75 28.80 28.80 28.85 28.85 28.95 28.95 29.30 29.30
 [49] 29.40 29.40 29.40 29.40 29.45 29.45 29.50 29.50 29.80 29.80 29.90 29.90
 [61] 30.00 30.00 30.05 30.05 30.10 30.10 30.50 30.50 30.55 30.55 30.60 30.60
 [73] 30.95 30.95 31.05 31.05 31.10 31.10 31.60 31.60 31.65 31.65 31.70 31.70
 [85] 32.25 32.25 32.75 32.75 33.90 33.90 36.40 36.40 37.80 37.80 38.65 38.65
 [97] 38.75 38.75 39.35 39.35 39.40 39.40 39.95 39.95 40.45 40.45 41.60 41.60
[109] 49.30 49.30
```

```
> ###################### COMPUTATION OF THE MEDIAN ############################
> # in this case  x  has length  n=10. There are 10 x 10 in the matrix +
> # 10  more for the extra diagonal. the median entry is the average of the
> # 55th and 56th, namely  (29.5+29.5)/2=29.5.
> median(c(.5*outer(x,x,"+"),x))
[1] 29.5
>
> ############## BOOTSTRAP BY SAMPLING FROM THIS DATA ######################
> # simulate Sampling Distribution
> B <- 10000
> v <- replicate(B,hl(sample(x,n,replace=T)))
> summary(v)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  25.75   28.80   29.50   29.88   30.55   41.60
> sd(v)
[1] 1.715152

> hist(v, col = rainbow(90), xlab = "Hodges-Lehman Estimator", freq = F,
+ main = paste("Bootsrap Std. Error, reps=", B, " samp.size=", n), breaks ="fd")
>  # M3074OxyCons2.pdf
>
> ############## SIMULATE BY SAMPLING FROM T(5) ##############################
>
> # samle from dist symmetric about zero
> B <- 10000
> n <- 10
> v <- replicate(B,hl(rt(n,5)))
> summary(v)
     Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
-1.718000 -0.244800  0.001588  0.002746  0.252100  1.584000
> sd(v)
[1] 0.3762987
>
> hist(v, col = rainbow(90), xlab = "Hodges-Lehman Estimator", freq = F,
+ main = paste("Simulate Std. Error, reps=", B, " samp.size=", n,
+ " from T(5)"), breaks = "fd")
> # M3074OxyCons3.pdf
```


Bootsrap Std. Error, reps= 10000 samp.size= 10
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
25
30
35
40
Hodges-Lehman Estimator


Simulate Std. Error, reps= 10000 samp.size= 10 from T(5)
Density
0.0
0.2
0.4
0.6
0.8
1.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
Hodges-Lehman Estimator