Math 3070 § 1. | **Paired vs. Pooled Example** | Name: Example
Treibergs | **Simulation to Compare Power** | July 30, 2011

This is Problem 6.15.10 of Navidi, *Statistics for Engineers and Scientists, 2nd ed.,* Mc Graw Hill, 2008. We wish to compare two t-tests for the equality of population means. For two samples, we consider the paired t-test with the pooled t-test.

Assume that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are two samples with means μ_X and μ_y, respectively. In case the readings X_i and Y_i are paired, such as those taken from different treatments from the same n individuals, we consider differences $D_i = X_i - Y_i$ and do a t test on the D_i's. In this case, under the null hypothesis $\mathcal{H}_0 : \mu_X = \mu_Y$, the statistic is

$$T_P = \frac{\overline{D}}{s_D/\sqrt{n}}$$

where $\overline{D}$ is the sample mean and s_D is the sample standard deviation of the D_i's. The variables X and Y may be dependent, but by considering the differences D_i we eliminate the variability due to individual differences. Assuming that $D_i \sim \mathrm{N}(\mu_D, \sigma_D)$ are a sample taken from a normal distribution, the statistic T_P sastisfies a t-distribution with $n-1$ degrees of freedom. Thus, at a significance of α, we reject the null hypothesis in favor of the alternative hypothesis $\mathcal{H}_a : \mu_X \neq \mu_Y$ if $|T_P| > t_{\alpha/2}$. The critical value is such that $\mathrm{P}(T > t_{\alpha/2}) = \alpha/2$ where T is distributed according to the t-distribution with $n-1$ degrees of freedom.

Assume on the other hand that $X_1, X_2, \ldots, X_n \sim \mathrm{N}(\mu_X, \sigma_X)$ and $Y_1, Y_2, \ldots, Y_n \sim \mathrm{N}(\mu_Y, \sigma_Y)$ are two independent normal samples such that $\sigma_X = \sigma_Y$. In this case, the data may be pooled to get a better estimator for $\sigma_X = \sigma_Y$

$$s_P^2 = \sqrt{\frac{(n-1)s_X^2 + (n-1)s_Y^2}{n+n-2}} = \sqrt{\frac{s_X^2 + s_Y^2}{2}}$$

where s_x^2 and s_Y^2 are the variances of the two samples. This time, under the null hypothesis $\mathcal{H}_0 : \mu_X = \mu_Y$, the statistic is

$$T_U = \frac{\overline{X} - \overline{Y}}{s_p\sqrt{\dfrac{1}{n} + \dfrac{1}{n}}} = \frac{\sqrt{n}(\overline{X} - \overline{Y})}{\sqrt{s_X^2 + s_Y^2}}$$

which satisfies the t-distribution with $n+n-2$ degrees of freedom. Thus, at a significance of α, we reject the null hypothesis in favor of the alternative hypothesis $\mathcal{H}_a : \mu_X \neq \mu_Y$ if $|T_U| > t_{\alpha/2}$. Now the critical value is such that $\mathrm{P}(T > t_{\alpha/2}) = \alpha/2$ where T is distributed according to the t-distribution with $2n-2$ degrees of freedom.

Assume now that $\mathcal{H}_0$ fails. We shall suppose that $n = 8$ and that $\mu_X = 0$ and $\mu_1 = 1$. The power of the test is the probability that the test rejects the null hypothesis given that $\mathcal{H}_0$ is false, specifically $\mu_X - \mu_Y = -1$. So one minus the power is the probability of a type II error. There is a trade-off. The paired test has smaller degrees of freedom, thus $t_{\alpha/2}$ is larger and the null hypothesis is harder to reject. For $\alpha = .05$ for the paired test $t_{\alpha/2} = 2.364624$ and for the pooled test, $t_{\alpha/2} = 2.144787$. But if the variables X and Y are correlated, then s_P underestimates the standard deviation whereas the differences are smaller. it turns out T_U has less variability but $\mathcal{H}_0$ is harder to reject.

We consider $B = 10,000$ samples taken from both independent and correlated X_i and Y_j. For each we do the paired test and the pooled test. The independent samples X is taken from $\mathrm{N}(0,1)$ and Y is taken from $\mathrm{N}(1,1)$. To simulate dependent samples, we take independent samples X from $\mathrm{N}(0,1)$ and Y from $\mathrm{N}(0,1)$ and consider $Z = 1 + .8X + .6Y$. The correlation is

$$\rho_{XZ} = \frac{\mathrm{Cov(X,Z)}}{\sqrt{\mathrm{V}(Z)}\sqrt{\mathrm{V}(Z)}}$$

The mean is

$$\mathrm{E}(Z) = \mathrm{E}(1 + .8X + .6Y) = \mathrm{E}(1) + .8\mathrm{E}(X) + .6\mathrm{E}(Y) = 1 + 0 + 0.$$

Since X and Y are independent, the variance is

$$\mathrm{V}(Z) = \mathrm{V}(1 + .8X + .6Y) = (.8)^2\mathrm{V}(X) + (.6)^2\mathrm{V}(Y) = .64 + .36 = 1.$$

The covariance is

$$\begin{aligned}
\mathrm{Cov}(X, Z) &= \mathrm{E}(XZ) - \mathrm{E}(X)\mathrm{E}(Z) \\
&= \mathrm{E}(X + .8X^2 + .6XY) - \mathrm{E}(X)\mathrm{E}(Z) \\
&= \mathrm{E}(X) + .8\mathrm{E}(X^2) + .6\mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Z) \\
&= \mathrm{E}(X) + .8(\mathrm{V}(X) + \mathrm{E}(X)^2) + .6\mathrm{E}(X)\mathrm{E}(Y) - \mathrm{E}(X)\mathrm{E}(Z) \\
&= 0 + .8(1 + 0) + 0 - 0 \\
&= .8.
\end{aligned}$$

Thus $\rho_{XZ} = .8$.

To estimate the power, we generate B samples of size n, compute the statistic and count the number of times the test rejects the null hypothesis. Then the estimate for the power is count$/B$. Here is a table of our power estimates:

Power	Paired t-test	Pooled t-test
Independent Samples	0.4097	0.4723
Correlated Samples	0.9662	0.4487

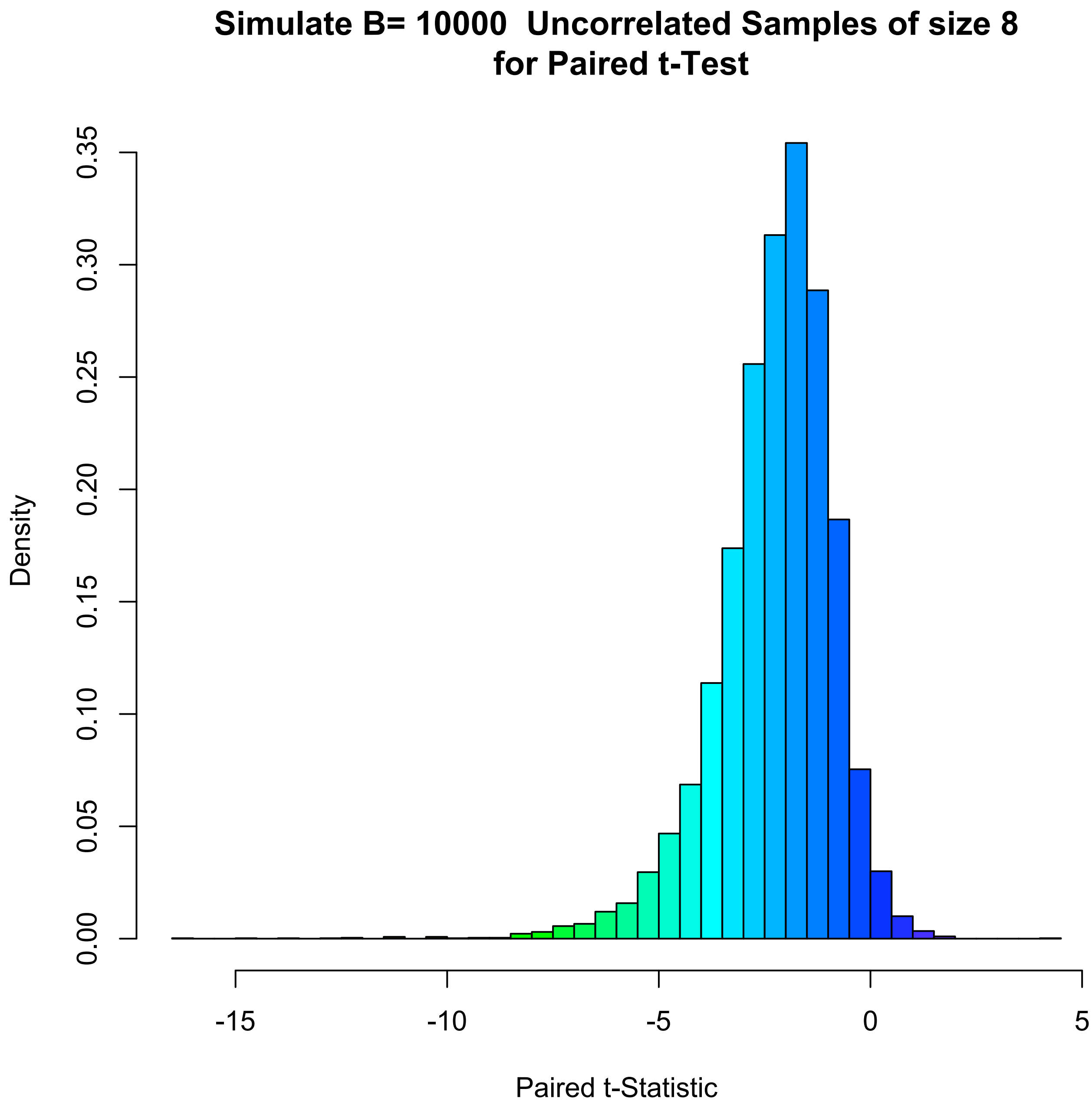
It worked out that for fixed $\mu_X - \mu_Y = -1$, for correlated data, the paired test had greater power and for independent data, the pooled t-test had better power.

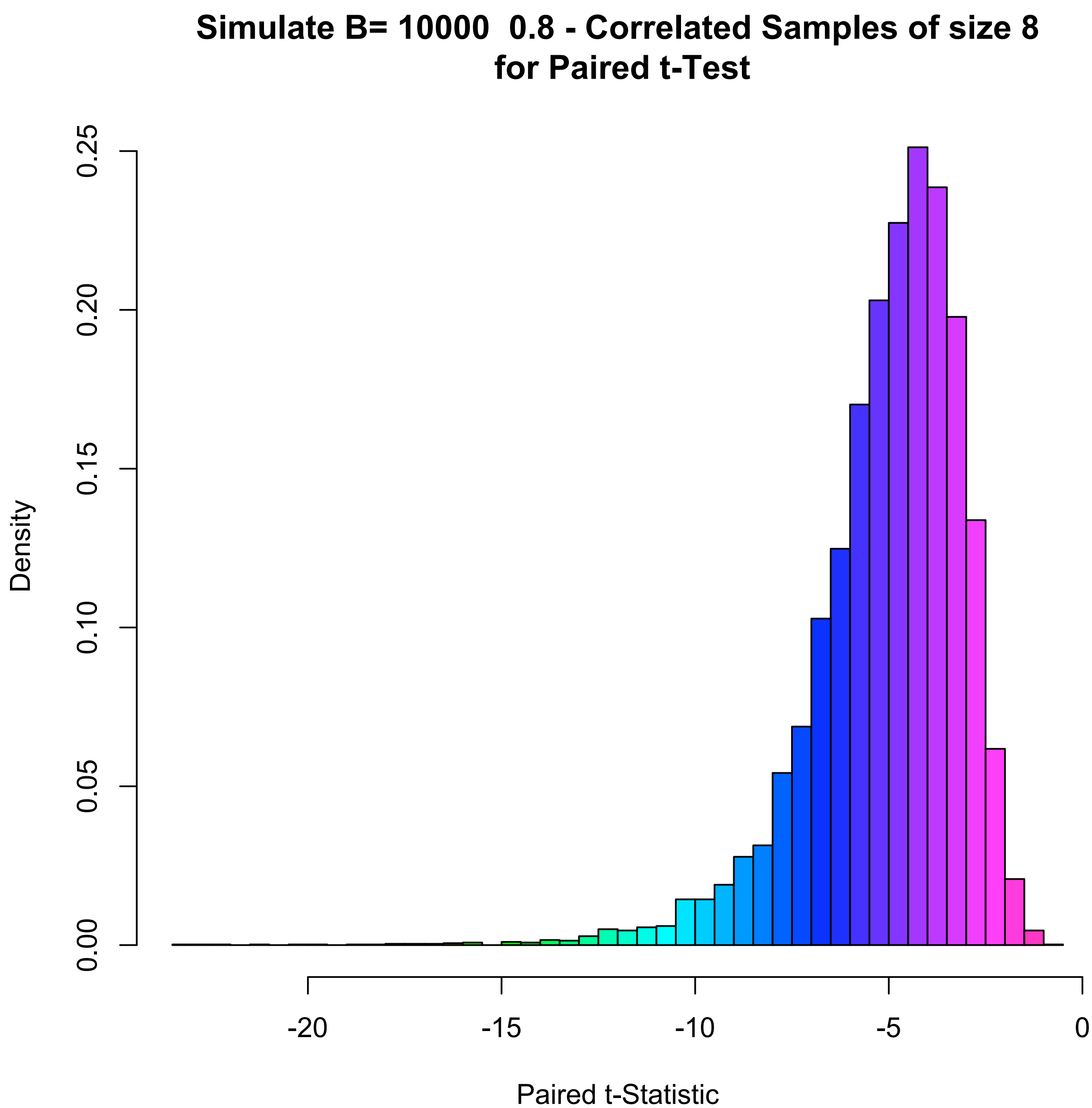
R Session:

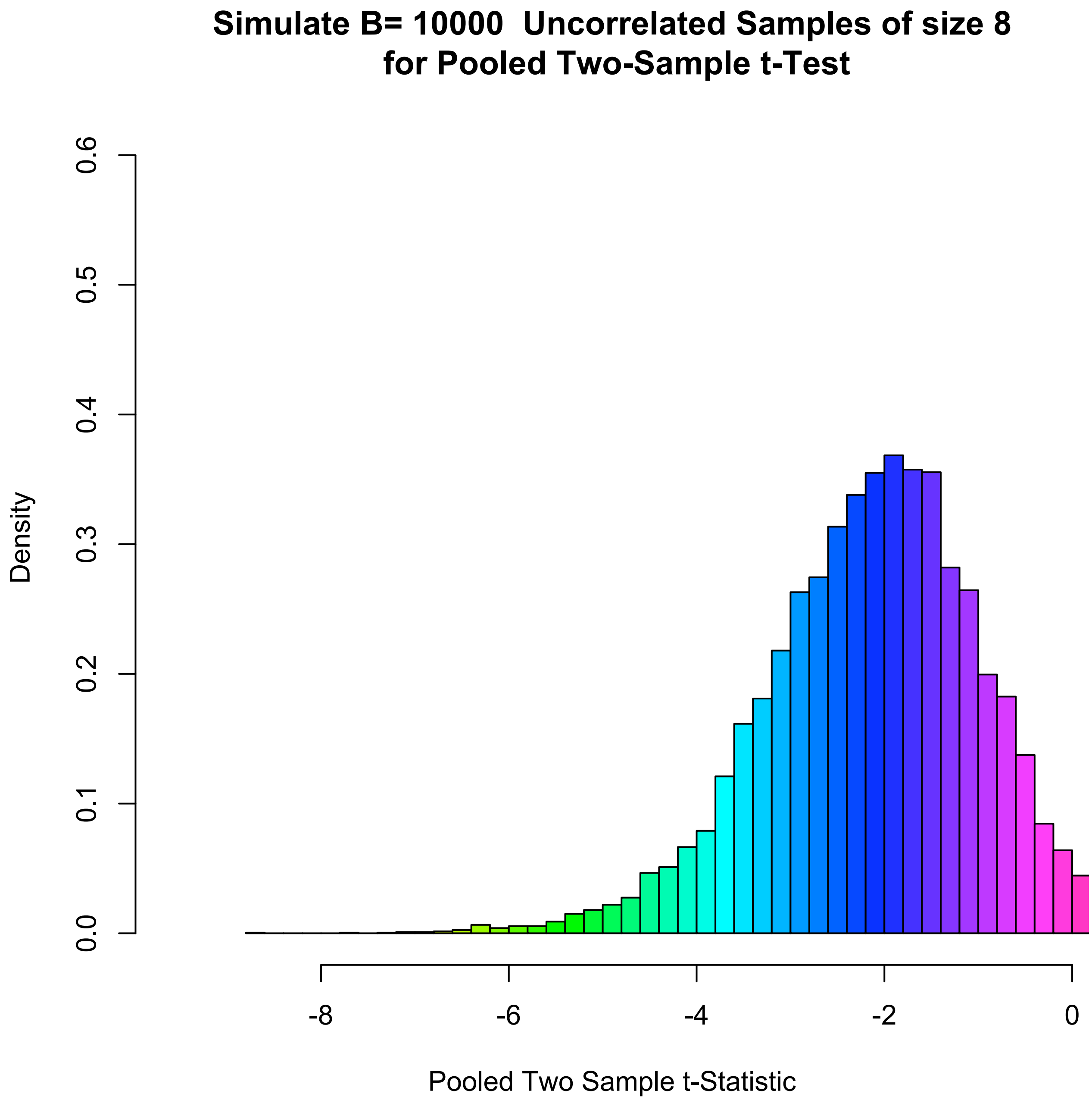
```
R version 2.10.1 (2009-12-14)
Copyright (C) 2009 The R Foundation for Statistical Computing
ISBN 3-900051-07-0

R is free software and comes with ABSOLUTELY NO WARRANTY.
You are welcome to redistribute it under certain conditions.
Type 'license()' or 'licence()' for distribution details.

  Natural language support but running in an English locale

R is a collaborative project with many contributors.
Type 'contributors()' for more information and
'citation()' on how to cite R or R packages in publications.

Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.
```

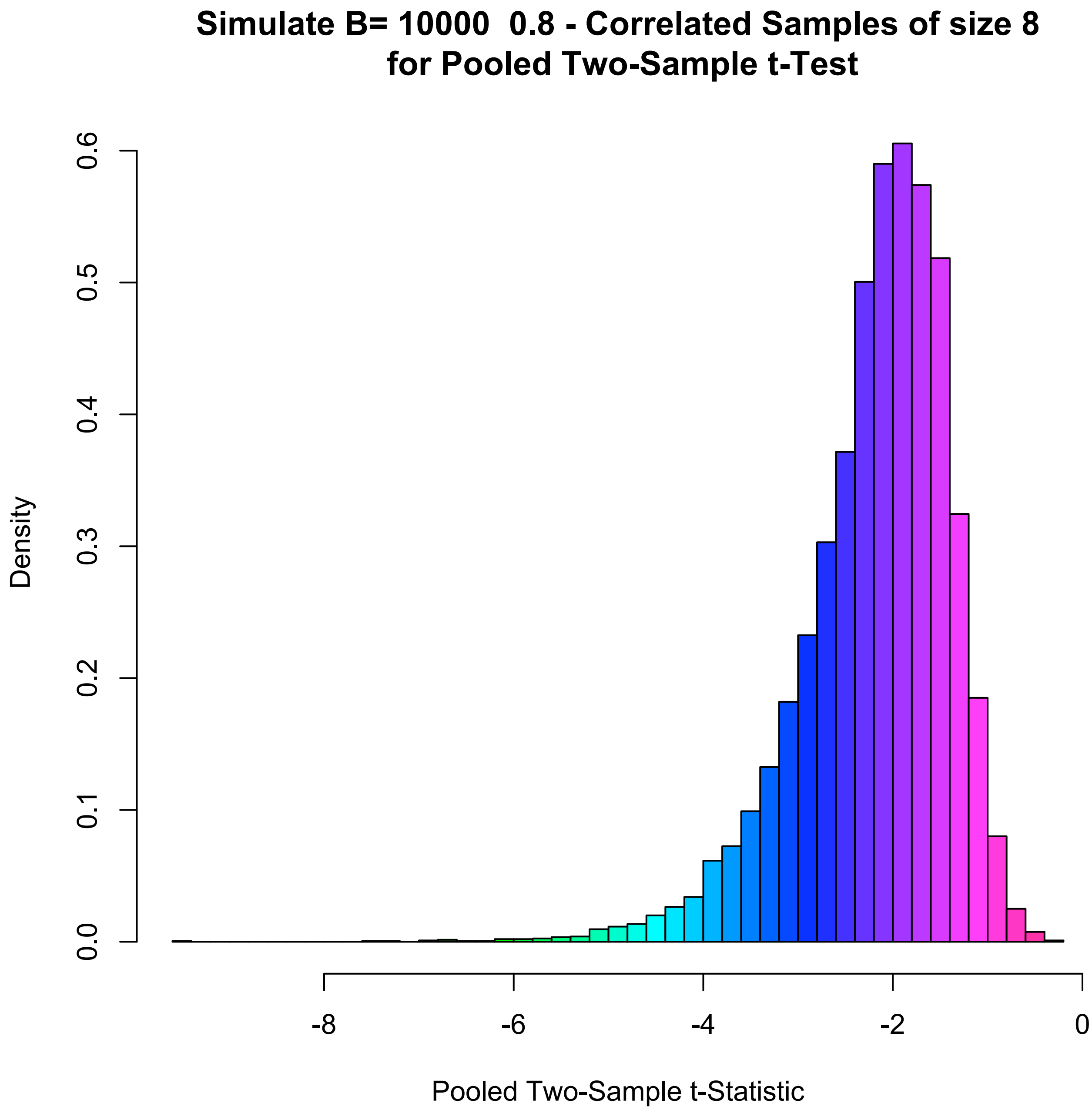
```
[R.app GUI 1.31 (5538) powerpc-apple-darwin8.11.1]

[Workspace restored from /Users/andrejstreibergs/.RData]

> ##### t-TEST STATISTIC FOR PAIRED DATA #######################################

> s8 <- sqrt(8)
> tp <- function(x,y){s8* mean(x-y)/sd(x-y)}
> B <- 10000
> alpha <- .05
> ta2 <- qt(alpha/2,7,lower.tail=F); ta2
[1] 2.364624

> # Estimate power for inderpendent data and paired t-test
> t<-replicate(B, tp(rnorm(8, 0, 1), rnorm(8, 1, 1)))
> sum(abs(t)>ta2)/B
[1] 0.4097
>
> hist(t, breaks = 40, col = rainbow(50), freq = F, main = paste(
+ "Simulate B=", B, " Uncorrelated Samples of size 8\n for Paired t-Test"),
+ xlab = "Paired t-Statistic")
> # M3074PowerPairing1.pdf
>
> ####### SAME TEST FOR CORRELATED DATA #########################################
> cp <- function(x,y){tp(x,1+.8*x+.6*y)}
> t<-replicate(B, cp(rnorm(8, 0, 1), rnorm(8, 0, 1)))
>
> # Estimate power for correlated data and paired t-test
> sum(abs(t)>ta2)/B
[1] 0.9662
>
> hist(t, breaks = 40, col = rainbow(50), freq = F, main = paste(
+ "Simulate B=", B, " 0.8 - Correlated Samples of size 8\n for
+ Paired t-Test"), xlab = "Paired t-Statistic")
> # M3074PowerPairing2.pdf
>
> ####### POOLED TEST FOR INDEPENDENT DATA #####################################
> tst <- 2*sqrt(2)
> tu <- function(x,y){tst*(mean(x)-mean(y))/sqrt(var(x)+var(y))}
> t <- replicate(B, tu(rnorm(8, 0, 1), rnorm(8, 1, 1)))
>
> tap2 <- qt(alpha/2,14,lower.tail=F); tap2
[1] 2.144787
> # Estimate power for independent data and pooled t-test
> sum(abs(t)>tap2)/B
[1] 0.4723

> hist(t, breaks = 40, col  = rainbow(50), freq = F, main = paste(
+ "Simulate B=", B, " Uncorrelated Samples of size 8\n for Pooled Two-Sample
+ t-Test"), xlab = "Pooled Two Sample t-Statistic")
> # M3074PowerPairing3.pdf
```

```
>
>
> ####### SAME TEST FOR CORRELATED DATA ########################################
> cup <- function(x,y){tu(x, 1+.8*x+.6*y)}
> t<-replicate(B, cup(rnorm(8, 0, 1), rnorm(8, 0, 1)))
>
> # Estimate power for dependent data and pooled t-test
> sum(abs(t)>tap2)/B
[1] 0.4487
>
> hist(t, breaks = 40, col = rainbow(50), freq = F, main = paste(
+ "Simulate B=", B, " 0.8 - Correlated Samples of size 8\n for Pooled
+ Two-Sample t-Test"), xlab = "Pooled Two-Sample t-Statistic")
> # M3074PowerPairing4.pdf
```


Simulate B= 10000 Uncorrelated Samples of size 8
for Paired t-Test
Density
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
-15
-10
-5
0
5
Paired t-Statistic


Simulate B= 10000 0.8 - Correlated Samples of size 8
for Paired t-Test
Density
0.00
0.05
0.10
0.15
0.20
0.25
-20
-15
-10
-5
0
Paired t-Statistic


Simulate B= 10000 Uncorrelated Samples of size 8
for Pooled Two-Sample t-Test
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
-8
-6
-4
-2
0
Pooled Two Sample t-Statistic


Simulate B= 10000 0.8 - Correlated Samples of size 8
for Pooled Two-Sample t-Test
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
-8
-6
-4
-2
0
Pooled Two-Sample t-Statistic