Math 3070 § 1. **First Midterm Exam** Name: Solutions
Treibergs $\sigma\tau$ September 18, 2013

(1.) *Web domain names consist of three or more characters which can be capital letters A–Z or numbers 0–9. Assuming that the first character must be a letter, how many three character web domain names are possible? Why? Of the three character domain names that start with a letter, how many are there that don't repeat a letter but may repeat a number? Why? (e.g., "B33" is counted but "BB3" is not.)*

Since the first character has to be a letter, there are $n_1 = 26$ possible choices, and the second and third characters may be either letters or numbers, there are $n_2 = n_3 = 36$ possible choices. By the multiplication principle, the number of three character web domain names is

$$n_1 \cdot n_2 \cdot n_3 = 26 \cdot 36 \cdot 36 = \boxed{33,696}$$

We count the cases LLL, LLN, LNL and LNN separately and add, since these are mutually exclusive, where L is letter and N is number. The name starts with a letter $n_1 = 26$. If there are to be no character duplications, then the second letter has one fewer choices since a letter has been taken and the third two fewer as two letters have been taken. Thus adding we find that the number of three character domain names that start with a letter and don't repeat letters is

$$\sharp\{LLL\}+\sharp\{LLN\}+\sharp\{LNL\}+\sharp\{LNN\} = 26\cdot25\cdot24+26\cdot25\cdot10+26\cdot10\cdot25+26\cdot10\cdot10 = \boxed{31,200}$$

(2.) *A standard deck of 52 cards consists of four suits $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. Each suit has 13 different kinds of cards $\{2, 3, 4, 5, 6, 7, 8, 9, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}, \mathrm{A}\}$. Suppose that a 13 card bridge hand is randomly dealt to you without replacement. How many different 13 card hands are there? Why? Let $\mathcal{A}$ be the event that your hand has two aces. What is the probability $P(\mathcal{A})$? Why? Let $\mathcal{B}$ be the event that your hand has all red cards (diamonds or hearts)? What is the probability $P(\mathcal{B})$? Why? What is the probability that your hand has two aces or has all red cards? Why? [You may leave answers in terms of binomial coefficients $\binom{p}{q}$.]*

We choose subsets of size 13 from 52, order not important, to find the number of hands

$$\binom{52}{13} = \boxed{635,013,559,600}$$

We select two aces from four possible aces and 11 remaining cards from the 48 non-aces where order is not important. Thus, assuming each hand is equally likely, the probability of getting two aces is

$$P(\mathcal{A}) = \frac{\sharp\{\text{hands with 2 aces}\}}{\sharp\{\text{hands}\}} = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} = \boxed{.2134934}$$

We select 13 cards from the 26 red cards, where order is not important. Thus, assuming each hand is equally likely, the probability of getting all red is

$$P(\mathcal{B}) = \frac{\sharp\{\text{hands with all red cards}\}}{\sharp\{\text{hands}\}} = \frac{\binom{26}{13}}{\binom{52}{13}} = \boxed{1.638 \times 10^{-5}}$$

Use the union formula. If all cards are red and there are two aces means that the diamond and heart aces are in the hand so the remaining 11 cards are chosen from the remaining 24 red cards, order is not important. Thus the probability that the hand has two aces or has all red cards is

$$P(\mathcal{A}\cup\mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B}) - P(\mathcal{A}\cap\mathcal{B}) = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} + \frac{\binom{26}{13}}{\binom{52}{13}} - \frac{\binom{24}{11}}{\binom{52}{13}} = \boxed{.2135058}$$

(3.) *In a Virginia Tech study about the relationship between hypertension and smoking habits, the following data was collected on 180 individuals. Suppose one respondent is chosen at random from this group. What is the probability that the respondent is experiencing hypertension? What is the probability that the respondent is neither experiencing hypertension nor is a heavy smoker? What is the conditional probability that the person is a non-smoker given that the person is experiencing no hypertension?*

	Non-smokers ($\mathcal{A}$)	Moderate smokers ($\mathcal{B}$)	Heavy smokers ($\mathcal{C}$)	Total
Hypertension ($\mathcal{H}$)	21	36	30	87
No hypertension ($\mathcal{H}'$)	48	26	19	93
Total	69	62	49	180

The events corresponding to classes of smokers, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are mutually exclusive and exhaustive so $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C} = \mathcal{S}$ where $\mathcal{S}$ is the sample space of all respondents. By the total probability formula, the number experiencing hypertension is the sum of number of non-smokers with hypertension plus the number of moderate smokers with hypertension plus the number of heavy smokers with hypertension. This number is divided by the total number of respondents assuming each respondent is equally likely

$$P(\mathcal{H}) = \frac{21 + 36 + 30}{180} = \frac{87}{180} = \boxed{.483}$$

The event that the respondent is not a heavy smoker $\mathcal{C}' = \mathcal{A} \cup \mathcal{B}$. Thus the event that the respondent is neither experiencing hypertension nor is a heavy smoker is $\mathcal{H}' \cap \mathcal{C}' = (\mathcal{H}' \cap \mathcal{A}) \cup (\mathcal{H}' \cap \mathcal{B})$. The probability

$$P(\mathcal{H}' \cap \mathcal{C}') = \frac{\sharp\{\mathcal{H}' \cap \mathcal{A}\} + \sharp\{\mathcal{H}' \cap \mathcal{B}\}}{\sharp\{\mathcal{S}\}} = \frac{48 + 26}{180} = \frac{74}{180} = \boxed{.411}$$

The conditional probability that the person is a non-smoker given that the person is experiencing no hypertension

$$P(\mathcal{A}|\mathcal{H}') = \frac{P(\mathcal{A} \cap \mathcal{H}')}{P(\mathcal{H}')} = \frac{48/180}{93/180} = \boxed{.516}$$

(4.) *Suppose that of all college graduates in Henefer, 45% graduated from the University of Utah, 30% graduated from Utah State, and 25% from BYU. Some graduates subscribe to the Wall Street Journal. Suppose that 20% of these University of Utah grads have a subscription to the Wall Street Journal, whereas the corresponding percentages for Utah State and BYU grads are 10% and 30%, respectively. What is the probability that a Henefer college grad graduated from the University of Utah and has a subscription to the Wall Street Journal? What is the probability that a that a Henefer graduate subscribes to the Wall Street Journal? Given that a Henefer graduate has a subscription to the Wall Street Journal, what is the probability that they graduated from the University of Utah?*

Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ denote the events that the Hennefer college graduate graduated from the University of Utah, Utah State and BYU, respectively. Let $\mathcal{W}$ be the event that the graduate subscribes to the Wall Street Journal. We are given $P(\mathcal{A}_1) = .45$, $P(\mathcal{A}_2) = .30$, $P(\mathcal{A}_3) = .30$ and the conditional probabilities $P(\mathcal{W}|\mathcal{A}_1) = .20$, $P(\mathcal{W}|\mathcal{A}_2) = .10$ and $P(\mathcal{W}|\mathcal{A}_1) = .30$. By the multiplication formula, the probability that a Henefer college grad graduated from the University of Utah and has a subscription to the Wall Street Journal is

$$P(\mathcal{A}_1 \cap \mathcal{W}) = P(\mathcal{A}_1)P(\mathcal{W}|\mathcal{A}_1) = (.45)(.20) = \boxed{.09}$$

By the total probability formula, the probability that a that a Henefer graduate subscribes to the Wall Street Journal is

$$\begin{aligned} P(\mathcal{W}) &= P(\mathcal{A}_1 \cap \mathcal{W}) + P(\mathcal{A}_2 \cap \mathcal{W}) + P(\mathcal{A}_3 \cap \mathcal{W}) \\ &= P(\mathcal{A}_1)P(\mathcal{W}|\mathcal{A}_1) + P(\mathcal{A}_2)P(\mathcal{W}|\mathcal{A}_2) + P(\mathcal{A}_3)P(\mathcal{W}|\mathcal{A}_3) \\ &= (.45)(.20) + (.30)(.10) + (.25)(.30) = \boxed{.195} \end{aligned}$$

The probability that a Henefer graduate graduated from the University of Utah given that they have a subscription to the Wall Street Journal is by the definition of conditional probability

$$P(\mathcal{A}_1|\mathcal{W}) = \frac{P(\mathcal{A}_1 \cap \mathcal{W})}{P(\mathcal{W})} = \frac{.09}{.195} = \boxed{.462}$$

(5.) *Three games manufactured by Elmo Puzzles Company are chosen at random. Let $\mathcal{A}_i$ denote the event that the ith game has a defect, where $i \in \{1, 2, 3\}$. Express the compound event "none of the games has a defect" in terms of the $\mathcal{A}_i$'s. Express $(\mathcal{A}_1 \cap \mathcal{A}_2)'$ in English. Express the compound event "at most one is defective" in terms of the $\mathcal{A}_i$'s. Express the compound event "exactly one is defective" in terms of the $\mathcal{A}_i$'s. Using Venn Diagrams or otherwise, show that $(\mathcal{A}_1 \cup \mathcal{A}_2) \cap \mathcal{A}_3' = (\mathcal{A}_1 \cap \mathcal{A}_3') \cup (\mathcal{A}_2 \cap \mathcal{A}_3')$.*

Since $\mathcal{A}_i'$ is the event that the ith game does not have a defect, the event "none of the games has a defect" may be rendered

$$\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3'.$$

The event $(\mathcal{A}_1 \cap \mathcal{A}_2)'$ may be expressed as

"It is not the case that both game 1 and game 2 are defective."

Because of de Morgan's rule $(\mathcal{A}_1 \cap \mathcal{A}_2)' = \mathcal{A}_1' \cup \mathcal{A}_2'$ we may say equivalently

"At least one of game 1 or game 2 is not defective."

The event "at most one is defective" means that none or exactly one is defective, which may be rendered

$$(\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3') \cup (\mathcal{A}_1 \cap \mathcal{A}_2' \cap \mathcal{A}_3') \cup (\mathcal{A}_1' \cap \mathcal{A}_2 \cap \mathcal{A}_3') \cup (\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3)$$

The event "exactly one is defective" means the first is defective and the others are not, or the second is defective and the others are not, or the third is defective and the others are not. It may be rendered

$$(\mathcal{A}_1 \cap \mathcal{A}_2' \cap \mathcal{A}_3') \cup (\mathcal{A}_1' \cap \mathcal{A}_2 \cap \mathcal{A}_3') \cup (\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3)$$

Using Venn Diagrams, we see the equlity $(\mathcal{A}_1 \cup \mathcal{A}_2) \cap \mathcal{A}_3' = (\mathcal{A}_1 \cap \mathcal{A}_3') \cup (\mathcal{A}_2 \cap \mathcal{A}_3')$ by

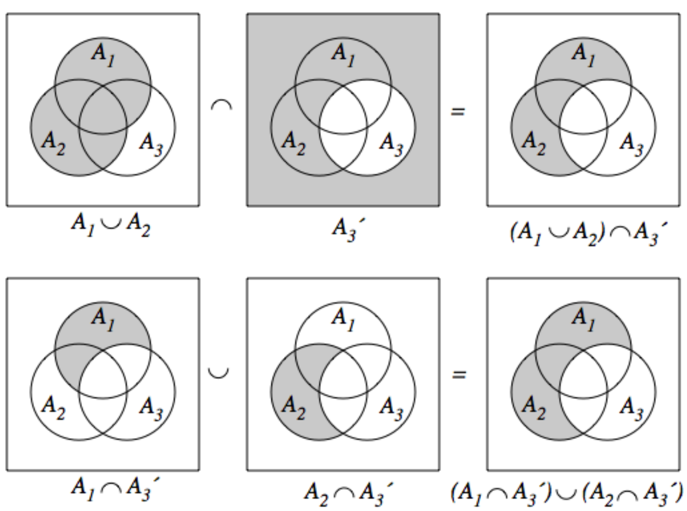

Figure 1: Venn Diagram for $(\mathcal{A}_1 \cup \mathcal{A}_2) \cap \mathcal{A}_3' = (\mathcal{A}_1 \cap \mathcal{A}_3') \cup (\mathcal{A}_2 \cap \mathcal{A}_3')$.