Math 3210 § 2. **Third Midterm Exam** Name: Solutions
Treibergs March 25, 2020

1. *Let $f(x)$ be defined on the reals. State the definition: $f(x)$ is continuous at the real number a. Prove that $f(x) = (1 + |x|)^2$ is continuous at a.*

   $f : \mathbf{R} \to \mathbf{R}$ is continuous at $a \in \mathbf{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $x \in \mathbf{R}$ and $|x - a| < \delta$.

   To see that f is continuous at a, choose $\epsilon > 0$. Let $\delta = \min\left\{1, \dfrac{\epsilon}{2 + 3|a|}\right\}$. For any $x \in \mathbf{R}$ such that $|x - a| < \delta$ we have by the triangle inequality

   $$|x| = |a + x - a| \le |a| + |x - a| < |a| + \delta \le |a| + 1.$$

   Furthermore this shows

   $$\begin{aligned} |f(x) - f(a)| &= |(1 + |x|)^2 - (1 + |a|)^2| \\ &= |(1 + |x| + 1 + |a|)(1 + |x| - 1 - |a|)| \\ &= (2 + |x| + |a|)\big||x| - |a|\big| \\ &\le (2 + |a| + 1 + |a|)|x - a| \\ &= (3 + 2|a|)|x - a| \\ &< (3 + 2|a|)\delta \\ &< (3 + 2|a|)\frac{\epsilon}{3 + 2|a|} = \epsilon. \end{aligned}$$

2. *Let the sequence $\{x_n\}$ be defined recursively by $x_1 = a$ where $0 < a < 1$ and $x_{n+1} = 2 + ax_n$. Prove that $\{x_n\}$ is bounded above. Prove that $\{x_n\}$ is strictly increasing. Is $\{x_n\}$ convergent? Why? If $x_n \to L$ as $n \to \infty$, what is L?*

   To see that $\{x_n\}$ is bounded above, we need to either guess the bound by iterating the recursion or see it from the graphs of $y = 2 + ax$ and $y = x$. The first few terms are

   $$a, 2 + a^2, 2 + 2a + a^3, 2 + 2a + 2a^2 + a^4, 2 + 2a + 2a^2 + 2a^3 + a^5, \ldots$$

   which seems to converge to $B = \sum_{k=0}^{\infty} 2a^k = \frac{2}{1-a}$. Oterwise, the recursion is a zig-zag path on the plane between the $y = 2 + ax$ and $y = x$ lines which intersect if $x = 2 + ax$ at $B = \frac{2}{1-a}$.

   We show that $\frac{2}{1-a}$ is an upper bound using induction. Base case $x_1 = a < 1 < 2 < \frac{2}{1-a}$. For the induction case, assume that for some n we have $x_n \le \frac{2}{1-a}$. Then since $a > 0$,

   $$x_{n+1} = 2 + ax_n \le 2 + a\frac{2}{1-a} = \frac{2 - 2a + 2a}{1 - a} = \frac{2}{1 - a}$$

   proving the induction case.

   We show that $\{x_n\}$ is strictly increasing by induction. For the base case, $x_2 = 2 + ax_1 = 2 + a^2 > 2 > 1 > a = x_1$. For the induction case, assume $x_{n+1} - x_n > 0$ for some n. Then

   $$x_{n+2} - x_{n+1} = 2 + ax_{n+1} - 2 - ax_n = a(x_{n+1} - x_n) > 0,$$

   since $a > 0$, proving the induction case.

Hence $\{x_n\}$ is a strictly increasing, bounded sequence. By the Monotone Convergence Theorem the sequence converges to a real number: there is $L \in \mathbf{R}$ such that $x_n \to L$ as $n \to \infty$. Taking limits of both sides of the recursion,

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (2 + ax_n) = 2 + aL.$$

Hence $L = \frac{2}{1-a}$.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *If $f : [0, 1] \to \mathbf{R}$ is continuous and one-to-one, then f is strictly monotone.*

   TRUE. Argue by contradiction. If f were not monotone, then there are three points $x_1 < x_2 < x_3$ in $[0, 1]$ such that either $f(x_1) < f(x_2) > f(x_3)$ or $f(x_1) > f(x_2) < f(x_3)$. ($f(x_1)$, $f(x_2)$, and $f(x_3)$ must be distinct since f is one-to-one.) In the former case, choose y such that $\max\{f(x_1), f(x_3)\} < y < f(x_2)$. In the latter case choose y such that $\min\{f(x_1), f(x_3)\} > y > f(x_2)$. In both cases, by the Intermediate Value Theorem applied to both $[x_1, x_2]$ and $[x_2, x_3]$ there are $c_1 \in (x_1, x_2)$ and $c_2 \in (x_2, x_3)$ such that $f(c_1) = y = f(c_2)$, contradicting f is one-to-one.

   (b) STATEMENT: *Let $I_1 \supset I_2 \supset I_3 \supset \cdots$ be a decreasing sequence of bounded intervals. Then the intersection is nonempty: $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.*

   FALSE. It does not specify closed intervals. Taking $I_n = (0, \frac{1}{n})$ gives decreasing sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \cdots$ but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

   (c) STATEMENT: *For $f : \mathbf{R} \to \mathbf{R}$ if $f(x_n) \to f(0)$ as $n \to \infty$ for some sequence such that $x_n \to 0$ as $n \to \infty$ then f is continuous at 0.*

   FALSE. The statement would be true if it said "for all sequences." To construct a counterexample, let

   $$f(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

   The sequence $x_n = \frac{1}{n} \to 0$ as $n \to \infty$ and $f(x_n) = 1 \to 1 = f(0)$ as $n \to \infty$ but f is not continuous at 0.

4. *Prove that if $f : \mathbf{R} \to \mathbf{R}$ is continuous at $a \in \mathbf{R}$ and $f(x) \geq 5$ for all $x \neq a$ then $f(a) \geq 5$.*

Choose $\epsilon > 0$. By the continuity of f at $a \in \mathbb{R}$, there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \qquad \text{whenever } x \in \mathbf{R} \text{ and } |x - a| < \delta.$$

Now choose any z close but not equal to a, such that $0 < |z - a| < \delta$. Hence for this z we have, using the assumption on $f(z)$,

$$f(a) = f(z) + f(a) - f(z) \geq f(z) - |f(a) - f(z)| > 5 - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude $f(a) \geq 5$.

5. *Define:* $\{S_n\}$ *is a* Cauchy Sequence. *Show that there is an* $L \in \mathbf{R}$ *such that* $S_n \to L$ *as* $n \to \infty$*, where*

$$S_n = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots + \frac{(-1)^{n+1}}{(2n)!} = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{(2k)!}.$$

$\{S_n\}$ is a *Cauchy Sequence* if for every $\epsilon > 0$ there is an $N \in \mathbf{R}$ such that $|S_\ell - S_m| < \epsilon$ whenever $\ell > N$ and $m > N$.

We show that $\{S_n\}$ is a Cauchy Sequence. Hence it is convergent: there is an $L \in \mathbf{R}$ such that $S_n \to L$ as $n \to \infty$. To see that $\{S_n\}$ is a Cauchy Sequence, choose $\epsilon > 0$. Let $N = \dfrac{\log\left(\frac{1}{\epsilon}\right)}{\log 2}$. Then for any $m, \ell > N$ we may suppose $\ell \geq m$. If $m = \ell$ then $|S_m - S_\ell| = 0 < \epsilon$. If $\ell > m$ we have

$$\begin{aligned}
|S_\ell - S_m| &= \left| \sum_{k=0}^{\ell} \frac{(-1)^{k+1}}{(2k)!} - \sum_{k=0}^{m} \frac{(-1)^{k+1}}{(2k)!} \right| \\
&= \left| \sum_{k=m+1}^{\ell} \frac{(-1)^{k+1}}{(2k)!} \right| \\
&\leq \sum_{k=m+1}^{\ell} \left| \frac{(-1)^{k+1}}{(2k)!} \right| \\
&= \sum_{k=m+1}^{\ell} \frac{1}{(2k)!} \\
&\leq \sum_{k=m+1}^{\ell} \frac{1}{2^k} \\
&= \frac{1}{2^{m+1}} \sum_{k=0}^{\ell-m-1} \frac{1}{2^k} \\
&= \frac{1}{2^{m+1}} \cdot \frac{1 - \left(\frac{1}{2}\right)^{\ell-m}}{1 - \frac{1}{2}} \\
&\leq \frac{1}{2^{m+1}} \cdot \frac{1}{\frac{1}{2}} = \frac{1}{2^m} \\
&< \frac{1}{2^N} = \epsilon
\end{aligned}$$

showing $\{S_n\}$ is a Cauchy Sequence. Here we have used for $k \geq m + 1 \geq 1$,

$$(2k)! = 1 \cdot 2 \cdot 3 \cdots (2k-1) \cdot 2k \geq 1 \cdot \overbrace{2 \cdot 2 \cdot \ldots \cdot 2}^{2k-1 \text{ factors}} \geq 2^k.$$